MATH 63CM MIDTERM EXAM SOLUTIONS

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1. Let $F : \mathbf{R}^n \to \mathbf{R}^n$ be a vectorfield such that

$$|F(p) - F(q)| \le K|p - q|$$

for all p and q (where $K < \infty$). Suppose $x(\cdot)$ and $y(\cdot)$ are solutions to the associated ODE (so that $x'(t) = F(x(t))$ and $y'(t) = F(y(t))$.) For $t \ge 0$, derive a bound for $|x(t) - y(t)|$ in terms of $|x(0) - y(0)|$. That is, find a function $G(t, d)$ so that

$$|x(t) - y(t)| \le G(t, |x(0) - y(0)|).$$

Solution: Let $u(t) = |x(t) - y(t)|^2$. Then

$$\begin{aligned} u'(t) &= \frac{d}{dt}((x(t) - y(t)) \cdot (x(t) - y(t))) \\ &= 2(x(t) - y(t)) \cdot (x'(t) - y'(t)) \\ &= 2(x(t) - y(t)) \cdot (F(x(t)) - F(y(t))) \\ &\le 2|x(t) - y(t)||F(x(t)) - F(y(t))| \\ &\le 2K|x(t) - y(t)|^2 \\ &= 2Ku(t). \end{aligned}$$

Thus

$$u' - 2Ku \le 0.$$

Multiplying by the integrating factor e^{-2Kt} gives

$$(e^{-2Kt}u)' \le 0,$$

so $e^{-2Kt}u(t) \le u(0)$, i.e. (multiplying by e^{2KT}),

$$|x(t) - y(t)|^2 \le e^{2Kt}|x(0) - y(0)|^2,$$

or, equivalently,

$$|x(t) - y(t)| \le e^{Kt}|x(0) - y(0)|.$$

2. Suppose A is an $n \times n$ complex matrix. Suppose $p(z)$ is a complex polynomial. Prove that μ is an eigenvalue of $p(A)$ if and only if $\mu = p(\lambda)$ for some eigenvalue λ of A. [Hint: You may wish first to consider the case that A is upper triangular.] [Hint: You may wish first to consider the case that A is upper triangular.]

Solution. **Case 1**: A is upper triangular with diagonal elements a_{ii}. Then A^k is also upper triangular with (i, i) entry a_{ii}^k. Thus $p(A)$ is upper triangular with (i, i) entry $p(a_{ii})$. Since the eigenvalues of an upper triangular matrix are precisely its diagonal entries, we have proved the assertion in the case of upper triangular matrices.

Case 2: arbitrary A. We know that there is an invertible matrix S such that $S^{-1}AS$ is upper triangular. By case 1, λ is an eigenvalue of $S^{-1}AS$ if and only if $p(\lambda)$ is an eigenvalue of $p(S^{-1}AS)$.

But A and $S^{-1}AS$ have the same eigenvalues. Also, $p(S^{-1}AS) = S^{-1}p(A)S$, so $p(A)$ and $p(S^{-1}AS)$ have the same eigenvalues. The result follows immediately.

If this is not clear, note that

$$
\begin{aligned}
\mu \text{ is an eigenvalue of } p(A) &\iff \mu \text{ is an eigenvalue of } S^{-1}p(A)S = p(S^{-1}AS) \\
&\iff \mu = p(\lambda) \text{ for some eigenvalue } \lambda \text{ of } S^{-1}AS \qquad \text{(by case 1)} \\
&\iff \mu = p(\lambda) \text{ for some eigenvalue } \lambda \text{ of } A.
\end{aligned}
$$

3. Suppose that U is an open subset of $\mathbf{R}^n$ and that $F : U \to \mathbf{R}^n$ is a C^1 vectorfield. Suppose also that

$$F(x) = A(\nabla V(x))$$

for some antisymmetric matrix $A \in \mathbf{R}^{n\times n}$ and some smooth function $V : U \to \mathbf{R}$.

(a). Show that if $x(\cdot)$ is a solution of $x' = F(x)$, then $V(x(t))$ is constant.

$$\begin{aligned} \frac{d}{dt} V(x(t)) &= \nabla V(x(t)) \cdot x'(t) \\ &= \nabla V(x(t) \cdot A \nabla V(x(t)) \\ &= 0 \end{aligned}$$

(since $\mathbf{v} \cdot A\mathbf{v} = 0$ for every $\mathbf{v} \in \mathbf{R}^n$ if A is antisymmetric.) Thus $V(x(t))$ is constant.

(b). Suppose V has a strict local minimum at $p \in U$. Why must p be a stable equilibrium?

This follows immediately from Lyapunov's Theorem. (We use V as the Lyapunov function.)

(c). Explain why p cannot be an asymptotically stable equilibrium.

For all sufficiently small r,

(*) $$V(x) > V(p)$$

for all $x \in \mathbf{B}_r(p) \setminus \{p\}$. Let $x(t)$ be the solution of $x' = Ax$ with $x(0) = x$. If $x(t) \to p$, then $V(x(t)) \to V(p)$, contradicting (*). Thus p is not asymptotically stable.

4. Suppose $F : K \to \mathbf{R}^n$ is a continuous vectorfield defined on a compact subset K of $\mathbf{R}^n$. Suppose $x_k(\cdot) : [0,1] \to K$ (for $k = 1, 2, \dots$) is a sequence of solutions of the ODE:

$$x_k'(t) = F(x_k(t)).$$

Prove that a subsequence $x_{k(i)}(\cdot)$ converges uniformly to a limit $x(\cdot) : [0,1] \to K$, and that $x'(t) = F(x(t))$.

Solution: Let $C = \max_{x \in K} F(x)$. (The maximum exists because F is continuous and K is compact.) Note that for $\tau \le t$,

$$\begin{aligned} |x_n(t) - x_n(\tau)| &= \left| \int_\tau^t x_n'(s)\, ds \right| \\ &= \left| \int_\tau^t F(x_n(s))\, ds \right| \\ &\le \int_\tau^t |F(x_n(s)|\, ds \\ &\le C|t - \tau| \end{aligned}$$

Thus the $x_n(\cdot)$ are all Lipschitz with the same Lipschitz bound C. By the Arzela-Ascoli Theorem, there is a subsequence $x_{k(i)}(\cdot)$ that converges uniformly to a continuous limit $x(\cdot) : [0,1] \to K$.

Claim: $F(x_{k(i)}(\cdot))$ converges uniformly to $F(x(\cdot))$.

Proof of claim: Since F is continuous on the compact set K, it is uniformly continuous. Thus for $\epsilon > 0$, there is a $\delta > 0$ such that $|p - q| < \delta \implies |F(p) - F(q)| < \epsilon$. By uniform convergence $x_{k(i)}(\cdot) \to x(\cdot)$, there is an N such that $i \ge N$, $t \in [0,1] \implies |x_{k(i)}(t) - x(t)| < \delta$. Therefore, $i \ge N$, $t \in [0,1] \implies |F(x_{k(i)}(t)) - F(x(t))| < \epsilon$. This proves the claim.

Note that

$$x_{k(i)}(t) - x_{k(i)}(0) = \int_0^t x_{k(i)}'(s)\, ds = \int_0^t F(x_{k(i)}(s))\, ds.$$

Letting $i \to \infty$ gives

$$\begin{aligned} x(t) - x(0) &= \lim_{i \to \infty} \int_0^t F(x_{k(i)}(s))\, ds \\ &= \int_0^t F(x(s))\, ds. \end{aligned}$$

(The second equality follows from uniform convergence.) By the fundamental theorem of calculus,

$$x'(t) = F(x(t)).$$

5. Find e^{At}, where $A = \begin{bmatrix} -1 & 6 \\ -2 & 6 \end{bmatrix}$. You may leave your answer in the form of the product of several matrices. [Hint: consider the vectors $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.]

Solution. Multiplying by A shows that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A with eigenvalues 2 and 3, respectively. Thus if $S = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ is the matrix with columns $\mathbf{v}_1$ and $\mathbf{v}_2$, we know that

$$S^{-1}AS = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

(We can also do the matrix multiplication to check this directly.) Consequently,

$$S^{-1}e^{tA}S = e^{tS^{-1}AS} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix},$$

so

(*) $$e^{tA} = S \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix} S^{-1}$$

or

(**) $$e^{tA} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

or

(***) $$e^{tA} = \begin{bmatrix} 4e^{2t} - 3e^{3t} & -6e^{2t} + 6e^{3t} \\ 2e^{2t} - 2e^{3t} & -3e^{2t} + 4e^{3t} \end{bmatrix}.$$

(Here, (*), (**), and (***) are all acceptable answers.)

6. Find e^{At} where $A = \begin{bmatrix} 4 & 4 \\ -1 & 0 \end{bmatrix}$.

Solution. $\det(\lambda I - A) = (\lambda - 4)(\lambda - 0) - (-4)(1) = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$.

Thus we know (from the general theory) that the matrix

$$N = A - 2I = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$$

is nilpotent; indeed, we know that $N^2 = 0$. (Of course we can also check directly that $N^2 = 0$.) Thus

$$\begin{aligned} e^{At} &= e^{2It+(A-2I)t} \\ &= e^{2It+Nt} \\ &= e^{2It}e^{Nt} \\ &= e^{2t}I(I + Nt) \\ &= e^{2t}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}\right) \\ &= e^{2t}\begin{bmatrix} 1+2t & 4t \\ -t & 1-2t \end{bmatrix}. \end{aligned}$$