Math 63CM Spring 2019 Homework 2 Solutions

1 Let $U = (x, u(t))$, so U satisfies the initial value problem

$$U'(t) = (0, F(U(t)))$$
$$U(0) = (x, y).$$

Note that the map $U \mapsto (0, F(U(t)))$ is locally Lipschitz if and only if F is, and also is C^1 if and only if F is. Therefore, the continuity of the map $(t, x, y) \mapsto \phi_t(x, y)$ follows from the continuity of flows with respect to time and initial conditions, which was proved in class. Moreover, if F is C^1, then the fact that the map $(t, x, y) \mapsto \phi_t(x, y)$ is C^1 follows from the continuous differentiability offlows of C^1 vector fields with respect to time and initial conditions, which was also proved in class.

2

a If λ is an eigenvalue of A^*A, then we have a nonzero vector x so that $A^*Ax = \lambda x$, so we have

$$0 \le |Ax|^2 = x^*A^*Ax = x^*\lambda x = \lambda |x|^2$$

so $\lambda \ge 0$.

b Let x_1 be such that $|x_1| = 1$ and $|Ax_1| = \max_{|x|=1} |Ax_1| = \|A\|_{\text{op}}$. Thus we must have, for every $y \perp x_1$, that

$$\lim_{\varepsilon \to 0} \frac{|A(x_1 + \varepsilon y)|^2 - |Ax_1|^2}{\varepsilon} = 0,$$

so

$$y^*A^*Ax_1 = 0,$$

which means that A^*Ax_1 must be a scalar multiple λ_1 of x_1, so λ_1 is an eigenvalue of A^*A. But then we have that

$$\|A\|_{\text{op}}^2 = |Ax_1|^2 = x_1^*A^*Ax_1 = x_1^*\lambda_1 x_1 = \lambda_1 |x_1|^2 = \lambda_1,$$

so the square root of largest eigenvalue of A^*A is at least $\|A\|_{\text{op}}$. But on the other hand, if λ is an eigenvalue of A^*A with eigenvector x, normalized so that $|x| = 1$, then we have

$$\|A\|_{\text{op}}^2 \ge |Ax|^2 = x^*A^*Ax = \lambda |x|^2 = \lambda,$$

so the square root of the largest eigenvalue of A^*A is at most $\|A\|_{\text{op}}$.

3

a Since $\sum_{n=0}^{\infty} a_n z^n$ converges, we must have $M = \sup_n |a_n z^n| < \infty$. This means that $|a_n| \le M|z|^{-n}$ for each $n \in \mathbf{N}$. Therefore, we have, whenever $0 \le r < |z|$,

$$\sum_{n=0}^{\infty} |a_n| r^n \le M \sum_{n=0}^{\infty} \left(\frac{r}{|z|} \right)^n < \infty$$

since $r/|z| < 1$.

b We recall that $\|A^n\|_{\text{op}} \le \|A\|_{\text{op}}^n$. Thus we have that

$$\sum_{n=0}^{\infty} \|a_n A^n\|_{\text{op}} \le \sum_{n=0}^{\infty} |a_n| \|A\|_{\text{op}}^n < \infty$$

by part (a). Therefore, the sequence $\sum_{n=0}^{\infty} a_n A^n$ converges absolutely, so it converges since the space of complex $n \times n$ matrices is a complete metric space.

4

(a) Notice that, for all $n \in \mathbf{N}$, $t, s \in [0, T]$, we have

$$|x_n(t) - x_n(s)| \le |t - s| \sup_{r \in [0,T]} |x'(r)| \le |t - s| \sup_{y \in K} |F(y)|,$$

and the last supremum is a finite number because F is continuous and K is compact. Therefore, the family (x_n) is uniformly Lipschitz, hence equicontinuous. Moreover, the family (x_n) is uniformly bounded since K is a compact subset of $\mathbf{R}^N$ hence bounded. By the Arzelà–Ascoli theorem, we thus have a subsequence $n(i)$ and a limit $x : [0, T] \to K$ so that $x_{n(i)}$ converges uniformly to x. Therefore, $x'_{n(i)} = F \circ x_{n(i)}$ converges uniformly to $F \circ x$. We recall that if $f_n \to f$ and $f'_n \to g$ uniformly, then $f' = g$. This implies that $x' = F \circ x$.

(b) Suppose that $x_i \in \overline{\mathbf{B}(p, R/3)}$ and that $x_i \to x$. By part (a), there is a subsequence (i_k) and a function $q : [0, \delta] \to \overline{\mathbf{B}(p, R)}$ so that $\phi(\cdot, x_{i_k}) \to q$ as $k \to \infty$ uniformly on $[0, \delta]$ and that $q'(t) = F(q(t))$ for $t \in [0, \delta]$. The uniform convergence of $\phi(\cdot, x_{i_k})$ to q means in particular that $q(0) = \lim_{k\to\infty} \phi(0, x_{i_k}) = \lim_{k\to\infty} x_{i_k} = x$. This means that q is the unique solution to the initial value problem

$$\begin{aligned} q'(t) &= F(q(t)) \\ q(0) &= 0. \end{aligned}$$

This means that if $((t_i, x_i))$ is a sequence in $[0, \delta] \times \overline{\mathbf{B}(p, R/3)}$, then for any subsequence (i_ℓ), we have a sub-sub-sequence (i_{ℓ_m}) so that $\phi(t_{i_{\ell_m}}, x_{i_{\ell_m}}) \to q(t, x)$. This implies that $\phi(t_i, x_i) \to q(t, x)$, which is what we wanted to show.

5

a We have

$$\frac{\mathrm{d}}{\mathrm{d}t}|x|^2 = \frac{\mathrm{d}}{\mathrm{d}t} x^t x = (x')^t x + x^t x' = x^t A^t x + x^t A x = x^t (A^t + A) x = 0$$

since $A(t)$ is antisymmetric. This means that $|x|^2$ is constant, so $|x|$ is constant.

b By the above computation, we have that

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}|x|^2 \bigg|_{t=0} = x(0)^t (A^t + A) x(0).$$

Since this must be true for every $x(0)$, this implies that $A^t + A = 0$, so A is antisymmetric.

6 We have

$$(D^2 - 5D + 6)u = 0,$$

which means that

$$(D - 3)(D - 2)u = 0.$$

Let $w = (D - 2)u$, so $(D - 3)w = 0$. This means that $Dw = 3w$, so $w = C_1 \mathrm{e}^{3t}$ for some constant C_1. Therefore, we have $(D - 2)u = C_1 \mathrm{e}^{3t}$, so

$$D(\mathrm{e}^{-2t} u) = -2\mathrm{e}^{-2t} u + \mathrm{e}^{-2t}(C_1 \mathrm{e}^{3t} + 2u) = C_1 \mathrm{e}^{t},$$

so

$$\mathrm{e}^{-2t} u = C_1 \mathrm{e}^{t} + C_2$$

for some constant C_2, so

$$u = C_1 \mathrm{e}^{3t} + C_2 \mathrm{e}^{2t}.$$

Note that

$$(D^2 - 5D + 6)(C_1 \mathrm{e}^{3t} + C_2 \mathrm{e}^{2t}) = 9C_1 \mathrm{e}^{3t} + 4C_2 \mathrm{e}^{2t} - 15C_1 \mathrm{e}^{3t} - 10C_2 \mathrm{e}^{2t} + 6C_1 \mathrm{e}^{3t} + 6C_2 \mathrm{e}^{2t} = 0,$$

so any u of this form is actually a solution. Therefore, the set of all solutions is

$$\{C_1 \mathrm{e}^{3t} + C_2 \mathrm{e}^{2t} \mid C_1, C_2 \in \mathbf{R}\}.$$