Math 63CM Spring 2019 Homework 3 Solutions

1 We have

$$0 = u'' - 4u' + u = D^2u - 4Du + u = (D - 2 + \sqrt{3})(D - 2 - \sqrt{3})u.$$

Let $v = (D - 2 - \mathrm{i}\sqrt{3})u$. Thus we have $Dv = (2 - \sqrt{3})v$, so $v = C_1 \mathrm{e}^{(2-\sqrt{3})t}$ for some constant C_1. This means that $(D - 2 - \sqrt{3})u = C_1 \mathrm{e}^{(2-\sqrt{3})t}$. Thus we have

$$\begin{aligned} D(\mathrm{e}^{\gamma t}u) &= \gamma \mathrm{e}^{\gamma t}u + \mathrm{e}^{\gamma t}Du \\ &= \gamma \mathrm{e}^{\gamma t}u + \mathrm{e}^{\gamma t}\left[C_1 \mathrm{e}^{(2-\sqrt{3})t} + (2 + \sqrt{3})u\right]. \end{aligned}$$

Take $\gamma = -(2 + \sqrt{3})$ to obtain

$$D(\mathrm{e}^{-(2+\sqrt{3})t}u) = C_1 \mathrm{e}^{-2\sqrt{3}t},$$

so

$$\mathrm{e}^{-(2+\sqrt{3})t}u = C_2 + C_1 \int_0^t \mathrm{e}^{-2\sqrt{3}s}\,\mathrm{d}s = C_2 + C_1\left(-\frac{1}{2\sqrt{3}s}\left(\mathrm{e}^{-2\sqrt{3}s} - 1\right)\right) = K_1 + K_2 \mathrm{e}^{-2\sqrt{3}s}$$

for different constants K_1 and K_2. This means that

$$u = K_1 \mathrm{e}^{(2+\sqrt{3})t} + K_2 \mathrm{e}^{(2-\sqrt{3})t}.$$

The initial conditions mean that we must have $K_1 + K_2 = a$ and $(2 + \sqrt{3})K_1 + (2 - \sqrt{3})K_2 = b$, which can be solved by

$$\begin{aligned} K_1 &= \frac{b - a(2 - \sqrt{3})}{2\sqrt{3}} \\ K_2 &= \frac{a(2 + \sqrt{3}) - b}{2\sqrt{3}}. \end{aligned}$$

So we have

$$u = \left(\frac{b - a(2 - \sqrt{3})}{2\sqrt{3}}\right)\mathrm{e}^{(2+\sqrt{3})t} + \left(\frac{a(2 + \sqrt{3}) - b}{2\sqrt{3}}\right)\mathrm{e}^{(2-\sqrt{3})t}.$$

2 We can rewrite the given differential equation as

$$\begin{pmatrix} u' \\ u \end{pmatrix}' = \begin{pmatrix} 4 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u' \\ u \end{pmatrix}.$$

Therefore, we have

$$\begin{pmatrix} u' \\ u \end{pmatrix} = \exp\left\{t\begin{pmatrix} 4 & -1 \\ 1 & 0 \end{pmatrix}\right\}\begin{pmatrix} a \\ b \end{pmatrix}.$$

Thus we need to diagonalize the matrix

$$\begin{pmatrix} 4 & -1 \\ 1 & 0 \end{pmatrix}.$$

We solve the characteristic equation

$$(4 - \lambda)(-\lambda) + 1 = \lambda^2 - 4\lambda + 1,$$

which has roots $\lambda = 2 \pm \sqrt{3}$. To get the $(2 + \sqrt{3})$-eigenvector, we solve

$$\begin{pmatrix} 2 - \sqrt{3} & -1 \\ 1 & -2 - \sqrt{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0,$$

which is solved by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 - \sqrt{3} \end{pmatrix}.$$

To get the $(2-\sqrt{3})$- eigenvector, we solve

$$\begin{pmatrix} 2+\sqrt{3} & -1 \\ 1 & -2+\sqrt{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0,$$

which is solved by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2+\sqrt{3} \end{pmatrix}.$$

Thus we have

$$\begin{pmatrix} 4 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2-\sqrt{3} & 2+\sqrt{3} \end{pmatrix}\begin{pmatrix} 2+\sqrt{3} & \\ & 2-\sqrt{3} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2-\sqrt{3} & 2+\sqrt{3} \end{pmatrix}^{-1},$$

so

$$\exp\left\{t\begin{pmatrix} 4 & -1 \\ 1 & 0 \end{pmatrix}\right\} = \begin{pmatrix} 1 & 1 \\ 2-\sqrt{3} & 2+\sqrt{3} \end{pmatrix}\begin{pmatrix} e^{(2+\sqrt{3})t} & \\ & e^{(2-\sqrt{3})t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2-\sqrt{3} & 2+\sqrt{3} \end{pmatrix}^{-1}.$$

We have that

$$\begin{pmatrix} 1 & 1 \\ 2-\sqrt{3} & 2+\sqrt{3} \end{pmatrix} = \frac{1}{2\sqrt{3}}\begin{pmatrix} 2+\sqrt{3} & -1 \\ -2+\sqrt{3} & 1 \end{pmatrix},$$

so

$$\begin{aligned}\exp\left\{t\begin{pmatrix} 4 & -1 \\ 1 & 0 \end{pmatrix}\right\} &= \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 1 \\ 2-\sqrt{3} & 2+\sqrt{3} \end{pmatrix}\begin{pmatrix} e^{(2+\sqrt{3})t} & \\ & e^{(2-\sqrt{3})t} \end{pmatrix}\begin{pmatrix} 2+\sqrt{3} & -1 \\ -2+\sqrt{3} & 1 \end{pmatrix} \\ &= \frac{1}{2\sqrt{3}}\begin{pmatrix} (-2+\sqrt{3})e^{(2-\sqrt{3})t}+(2+\sqrt{3})e^{(2+\sqrt{3})t} & e^{(2-\sqrt{3})t}-e^{(2+\sqrt{3})}t \\ -e^{(2-\sqrt{3})t}+e^{(2+\sqrt{3})}t & (2+\sqrt{3})e^{(2-\sqrt{3})t}-(2-\sqrt{3})e^{(2+\sqrt{3})t} \end{pmatrix}.\end{aligned}$$

Therefore, the solution is given by the second coordinate of $\exp\left\{t\begin{pmatrix} 4 & -1 \\ 1 & 0 \end{pmatrix}\right\}\begin{pmatrix} b \\ a \end{pmatrix}$, which is

$$\begin{aligned} u &= \left(\frac{-e^{(2-\sqrt{3})t}+e^{(2+\sqrt{3})}t}{2\sqrt{3}}\right)b + \left(\frac{(2+\sqrt{3})e^{(2-\sqrt{3})t}-(2-\sqrt{3})e^{(2+\sqrt{3})t}}{2\sqrt{3}}\right)a \\ &= \frac{b-(2-\sqrt{3})a}{2\sqrt{3}}e^{(2+\sqrt{3})t} + \frac{(2+\sqrt{3})a-b}{2\sqrt{3}}e^{(2-\sqrt{3})t},\end{aligned}$$

which, amazingly enough, matches what we got in part (a).

3 Since A is upper-triangular, the eigenvalues of A are the diagonal entries -1 and 2. The corresponding eigenvectors are $(1,0)$ and $(1,1)$, respectively. Therefore, we can write

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & \\ & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} -1 & \\ & 2 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ & 1 \end{pmatrix},$$

so

$$e^{At} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{-t} & \\ & e^{2t} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{-t} & -e^{-t} \\ & e^{2t} \end{pmatrix} = \begin{pmatrix} e^{-t} & -e^{-t}+e^{2t} \\ & e^{2t} \end{pmatrix}.$$

4 Since A is real, its char poly is also real, and so all non-real roots of the char poly must come in conjugate pairs, so ci and $-c$i must both be eigenvalues. Let v be an eigenvector for ci, so $Av = civ$. Then we have that $A\overline{v} = \overline{Av} = \overline{civ} = -civ$, so $\overline{v}$ is an eigenvector for $-c$i. Then $e^{At}(v+\overline{v})$ is of course a solution to $x' = Ax$, and also

$$e^{At}(v+\overline{v}) = \sum_{k=0}^{\infty}\frac{1}{k!}t^kA^k(v+\overline{v}) = \sum_{k=0}^{\infty}\frac{1}{k!}t^k(ci)^kv + \sum_{k=0}^{\infty}\frac{1}{k!}t^k(-ci)^k\overline{v} = e^{cit}v + e^{-cit}\overline{v} = 2\,\mathrm{Re}(e^{cit}v),$$

which is evidently real and $2\pi/c$-periodic in t. Note that the second inequality in the above display is justified because both of the sums in the third expression converge absolutely, so there is no problem rearranging and combining their terms to get the second expression.

5 We have a change-of-basis matrix C so that

$$A = C\begin{pmatrix} B_{\mathrm{i}} & \\ & B_{-\mathrm{i}} \end{pmatrix} C^{-1},$$

where $B_{\pm\mathrm{i}}$ are upper-triangular 2×2 matrices with $B_{\pm\mathrm{i}} = \pm\mathrm{i}I + N_{\pm\mathrm{i}}$, where $N_{\pm i}$ are nilpotent 2×2 matrices. Since A is not diagonalizable, at least one of $N_{\pm\mathrm{i}}$ must be nonzero. Assume without loss of generality that N_{i} is nonzero. (Otherwise, we can consider the complex conjugate of A.) Thus we have

$$N_{\mathrm{i}} = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}$$

for some $c \neq 0$. Then we have

$$\exp(At) = C\begin{pmatrix} \mathrm{e}^{tB_{\mathrm{i}}} & \\ & \mathrm{e}^{tB_{-\mathrm{i}}} \end{pmatrix} C^{-1},$$

and

$$\mathrm{e}^{tB_{\mathrm{i}}} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}t} & ct\mathrm{e}^{\mathrm{i}t} \\ & \mathrm{e}^{\mathrm{i}t} \end{pmatrix}.$$

Therefore,

$$\exp(At)C\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = C\begin{pmatrix} ct\mathrm{e}^{\mathrm{i}t} \\ \mathrm{e}^{\mathrm{i}t} \\ 0 \\ 0 \end{pmatrix},$$

and $|(ct\mathrm{e}^{\mathrm{i}t}, \mathrm{e}^{\mathrm{i}t}, 0, 0)| \to \infty$ as $t \to \infty$, so $|\exp(At)C(0,1,0,0)| = |C(ct\mathrm{e}^{\mathrm{i}t}, \mathrm{e}^{\mathrm{i}t}, 0, 0)| \to \infty$ as $t \to \infty$ as well, since C is invertible so $|(ct\mathrm{e}^{\mathrm{i}t}, \mathrm{e}^{\mathrm{i}t}, 0, 0)| = |C^{-1}C(ct\mathrm{e}^{\mathrm{i}t}, \mathrm{e}^{\mathrm{i}t}, 0, 0)| \le \|C^{-1}\|_{\mathrm{op}}|C(ct\mathrm{e}^{\mathrm{i}t}, \mathrm{e}^{\mathrm{i}t}, 0, 0)|$. Thus, $x(t) = \exp(At)C(0,1,0,0)$ is a solution of $x'(t) = Ax(t)$ so that $|x(t)| \to \infty$ as $t \to \infty$.

6

a We have

$$\mathrm{tr}(CD) = \sum_i (CD)_{ii} = \sum_{i,j} C_{ij}D_{ji},$$

which is evidently symmetric in C and D, so $\mathrm{tr}(CD) = \mathrm{tr}(DC)$.

b We have $\mathrm{tr}(S^{-1}AS) = \mathrm{tr}(SS^{-1}A) = \mathrm{tr}(A)$ by part (a).

c We always have a matrix C so that $A = C^{-1}(D+N)C$, where D is diagonal, N is strictly upper-triangular, and D and N commute. This is a consequence of Theorem 6 in blocks.pdf (or of Jordan canonical form if you know about that). Thus we have

$$\det \mathrm{e}^A = \det(C^{-1}\exp(D+N)C) = \det(C)^{-1}\det(\exp(D))\det(\exp(N))\det C = \det(\exp(D))\det(\exp(N)).$$

We note that N^k for $k \ge 1$ is strictly upper-triangular, so $\exp(N)$ has 1s on the diagonal, so $\det(\exp(N)) = 1$. On the other hand, $\det(\exp(D)) = \prod_{j=1}^n \exp(D_{jj}) = \exp(\mathrm{tr}\, D)$. Therefore, $\det(\exp A) = \exp(\mathrm{tr}\, D)$. On the other hand, we have $\mathrm{tr}\, A = \mathrm{tr}(D+N) = \mathrm{tr}\, D$ since N is strictly upper triangular, so $\det(\exp(A)) = \exp(\mathrm{tr}\, A)$.

7 As in the previous problem, we have a matrix C so that $A = C^{-1}(D+N)C$, where D is diagonal, N is strictly upper-triangular, and D and N commute. Then the set of eigenvalues of A is exactly the entries of D. The set of eigenvalues of e^{tA} is exactly the set of eigenvalues of $\mathrm{e}^{t(D+N)} = \mathrm{e}^{tD}\mathrm{e}^{tN}$. We note that e^{tD} is a diagonal matrix and e^{tN} is an upper-triangular matrix with 1s on the diagonal, so $\mathrm{e}^{tD}\mathrm{e}^{tN}$ is an upper-triangular matrix whose diagonal entries, and hence eigenvalues, are the diagonal entries of e^{tD}, which form the set $\{\mathrm{e}^{t\lambda} \mid \lambda \text{ is an eigenvalue of } A\}$. This proves the result.

8 As in the last two problems, we have a matrix C so that $A = C^{-1}(D+N)C$, where D is diagonal, N is strictly upper-triangular, and D and N commute. Since all of the eigenvalues of A are strictly less than 1 in magnitude, we have $\|D\|_{\mathrm{op}} < 1$. Then we have that

$$A^k = C^{-1}(D+N)^k C.$$

Since D and N commute, we can write, by the binomial theorem,

$$(D+N)^k = \sum_{j=0}^{k} \binom{k}{j} D^{k-j} N^j.$$

On the other hand, we note that $N^j = 0$ whenever $j > n$. Thus we have, for $k \geq n$,

$$(D+N)^k = \sum_{j=0}^{n} \binom{k}{j} D^{k-j} N^j.$$

Thus,

$$\|(D+N)^k\|_{\mathrm{op}} \leq \sum_{j=0}^{n} \binom{k}{j} \|D\|_{\mathrm{op}}^{k-j} \|N\|_{\mathrm{op}}^{j} \leq nk^j \|D\|_{\mathrm{op}}^{k-n} \left(1 + \|N\|_{\mathrm{op}}\right)^n.$$

But since $\|D\|_{\mathrm{op}} < 1$, we have $\lim_{k\to\infty} k^j \|D\|_{\mathrm{op}}^{k-n} = 0$, so $\lim_{k\to\infty} \|(D+N)^k\|_{\mathrm{op}} = 0$, so $\lim_{k\to\infty}(D+N)^k = 0$.