ASYMPTOTIC BEHAVIOR OF $e^{At}v$: PART I

Lemma 1. *Suppose A in an $n \times n$ matrix with only one eigenvalue λ (i.e., only one root of the characteristic polynomial). Suppose $v \in \mathbf{C}^n$ is not zero. Then there is an integer k with $0 \le k \le n-1$ and a nonzero vector $c \in \mathbf{C}^n$ such that*

$$\lim_{t\to\pm\infty} \frac{e^A t v}{e^{\lambda t} t^k} = c. \tag{1}$$

Proof.

$$\begin{aligned} e^{At}v &= e^{\lambda t} p_{n-1}(Nt)v \\ &= e^{\lambda t}\left(I + Nt + \cdots + \frac{(Nt)^{n-1}}{(n-1)!}\right)v \\ &= e^{\lambda t}\left(v + (Nv)t + \cdots + \frac{N^{n-1}v}{(n-1)!}t^{n-1}\right) \\ &= e^{\lambda t}(c_0 + c_1 t + \ldots c_{n-1}t^{n-1}), \end{aligned}$$

where c_j the vector $c_j = \frac{1}{j!}N^j v$. Note that $c = 0 = v \neq 0$. Let k be the largest index such that $c_k \neq 0$. Then

$$e^{At}v = e^{\lambda t}(c_0 + c_1 t + \cdots + c_k t^k).$$

Thus

$$\frac{e^{At}v}{e^{\lambda t}t^k} = \frac{c_0}{t^k} + \frac{c_1}{t^{k-1}} + \cdots + c_k,$$

which clearly tends to c_k as $t \to \pm\infty$. □

Remark 1. The same proof shows that if A is any $n \times n$ matrix and if $v \neq 0$ is a generalized eigenvector of A with eigenvalue λ, then there is a nonzero vector c and in integer k such that (1) holds.

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Let $z \in C$. Then $z = a + bi$ where a and b are real. We say that $a = \mathrm{Re}(z)$ is the **real part** of z. Recall that $|a + bi|$ is defined to be $\sqrt{a^2 + b^2}$. Recall also that $e^{a+bi} = e^a(\cos b + i \sin b)$, so

$$|e^{a+bi}| = |e^a|\,(\cos^2 b + \sin^2 b) = e^a.$$

In other words,

$$|e^z| = e^{\mathrm{Re}(z)}$$

for any complex number z.

Consequently, Lemma 1 implies

$$\lim_{t\to\pm\infty} \frac{|e^{At}v|}{e^{\mathrm{Re}(\lambda)t}|t|^k} = |c| \neq 0. \tag{2}$$

Now suppose that $\mathrm{Re}(\lambda) > 0$. Then $e^{\mathrm{Re}(\lambda)t}|t|^k \to \infty$ as $t \to \infty$, so $|e^{At}v| \to \infty$ by (2).

As $t \to -\infty$, $e^{\mathrm{Re}(\lambda)t}|t|^k \to 0$. Thus by (2),

$$(*) \qquad \lim_{t\to-\infty} |e^{At}v| = 0.$$

In particular

$$\text{column } j \text{ of } e^{At} = e^{At}e_j \to 0 \quad \text{as } t \to -\infty.$$

Since this is true for each column, we see that $e^{At} \to 0$ as $t \to -\infty$.

Similar arguments apply when $\mathrm{Re}(\lambda) < 0$. Thus we have proved:

Theorem 2. *Suppose A is a matrix with only one characteristic root λ. If $\mathrm{Re}(\lambda) > 0$, then*

$$\lim_{t\to-\infty} e^{At} = 0.$$

and

$$\lim_{t\to+\infty} |e^{At}v| = \infty$$

for every nonzero $v \in \mathbf{C}^n$.

If $\mathrm{Re}(\lambda) < 0$, then

$$\lim_{t\to\infty} e^{At} = 0.$$

and

$$\lim_{t\to-\infty} |e^{At}v| = \infty$$

for every nonzero $v \in \mathbf{C}^n$.

1. More General Matrices

Theorem 3. *Let A be a square matrix with $\det(\lambda I - A) = \prod_{i=1}^k (\lambda - \lambda_i)^{\nu_i}$. where the $\lambda_1, \ldots, \lambda_k$ are distinct.*

(1) *If $\mathrm{Re}(\lambda_i) < 0$ for each i, then*

$$\lim_{t\to\infty} e^{At} = 0$$

and

$$\lim_{t\to-\infty} |e^{At}v| = \infty$$

for every nonzero $v \in \mathbf{C}^n$.

(2) *If $\mathrm{Re}(\lambda_i) > 0$ for each i, then*

$$\lim_{t\to\infty} e^{At} = 0$$

and

$$\lim_{t\to-\infty} |e^{At}v| = \infty$$

for every nonzero $v \in \mathbf{C}^n$.

Proof. We prove it in the case $\mathrm{Re}(\lambda_i) < 0$ for each i; the other case is proved in exactly the same way.

We know that $A = S^{-1}MS$ for a block diagonal matrix M where the ith block, M_i, is an upper-triangular $\nu_i \times \nu_i$ matrix with all eigenvalues $= \lambda_i$.

By Theorem 3, $\lim_{t\to\infty} e^{M_i t} = 0$ for each i.

Thus

$$e^{Mt} = \begin{bmatrix} e^{tM_1} & 0 & \dots & 0 \\ 0 & e^{tM_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{tM_k} \end{bmatrix} \to 0$$

as $t \to \infty$.

Hence

$$\lim_{t\to\infty} e^{At} = \lim_{t\to\infty}(S^{-1}e^{Mt}S) = S^{-1}(\lim_{t\to\infty} e^{Mt})S = 0.$$

Thus we have proved the first assertion of the theorem.

Now we analyze the situation as $t \to -\infty$. Let v be a nonzero vector in $\mathbf{C}^n$. Let $\mathbf{v}_1 \in \mathbf{C}^{\nu_1}$ be the vector whose entries are the first ν_1 entries of v. Let $\mathbf{v}_2 \in \mathbf{C}^{\nu_2}$ be the vector whose entries are the next ν_2 entries of v, and so on. Thus

$$v = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_k \end{bmatrix}$$

and

$$e^{Mt}v = \begin{bmatrix} e^{M_1t}\mathbf{v}_1 \\ e^{M_2t}\mathbf{v}_2 \\ \vdots \\ e^{M_kt}\mathbf{v}_k \end{bmatrix}.$$

Since $v \neq 0$, $\mathbf{v}_j \neq 0$ for some j. Thus

$$|e^{Mt}v| \geq |e^{M_jt}\mathbf{v}_j| \to \infty \quad \text{as } t \to -\infty \tag{3}$$

by Theorem 3.

This holds for each nonzero vector v. If $x \neq 0$, then $Sx \neq 0$. Therefore

$$\lim_{t\to-\infty} |e^{Mt}Sx| = \infty.$$

Now for any invertible matrix S and any vector v,

$$|S^{-1}v| \geq \frac{|v|}{\|S\|_{\mathrm{op}}}.$$

(See Lemma 2 below.) Thus

$$|e^{At}x| = |S^{-1}e^{Mt}Sx| \geq \frac{|e^{Mt}Sx|}{\|S\|_{\mathrm{op}}},$$

so

$$\lim_{t\to-\infty} |e^{At}v| = \infty$$

by (3). □

Lemma 2. *Suppose S is an invertible matrix. Then*

$$|S^{-1}v| \geq \frac{|v|}{\|S\|_{\mathrm{op}}}.$$

Proof. For any vector x,

$$|Sx| \leq \|S\|_{\mathrm{op}}|x|.$$

In particular, it holds for $x = S^{-1}v$:

$$|v| \leq \|S\|_{\mathrm{op}}|S^{-1}v|.$$

□