CALCULUS OF VARIATIONS

In calculus, one studies min-max problems in which one looks for a number or for a point that minimizes (or maximizes) some quantity. The calculus of variations is about min-max problems in which one is looking not for a number or a point but rather for a function that minimizes (or maximizes) some quantity.

For example: given two points (x_0, y_0) and (x_1, y_1), find the shortest curve (that is a graph) joining the two points. That is, find a function $y(\cdot) : [x_0, x_1] \to \mathbf{R}$ with $y(x_0) = y_0$ and $y(x_1) = y_1$ that makes the arclength

$$\mathcal{L}[y(\cdot)] = \int_{x_0}^{x_1} \sqrt{1 + \dot{y}^2}\, dx$$

as small as possible. (Here $\dot{y}$ denotes $y'(x) = \frac{dy}{dx}$.)

(If we let s denote arclength, then $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (\frac{dy}{dx})^2}\, dx$.)

More generally, given any C^2 function

$$L : \mathbf{R}^3 \to \mathbf{R},$$

we can look for a function $y(\cdot) : [x_0, x_1] \to \mathbf{R}$ that makes the quantity

$$\mathcal{L}[y(\cdot)] = \int_{x_0}^{x_1} L(x, y(x), \dot{y}(x))\, dx$$

as small as possible.

In general, the minimum might not exist. However, if the minimum does exist, then it has to satisfy a differential equation called the **Euler-Lagrange Equation**. If we can solve the Euler-Lagrange Equation, then we can find the minimum (if it exists.)

Theorem 1. *Suppose $y(\cdot) : [x_0, x_1] \to \mathbf{R}$ is a C^2 function that minimizes*

$$\mathcal{L}[y(\cdot)] = \int_{x_0}^{x_1} L(x, y(x), \dot{y}(x))\, dx$$

subject to the boundary conditions $y(x_0) = y_0$ and $y(x_1) = y_1$. Then $y(\cdot)$ is a solution to the differential equation

$$(*) \qquad \frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0.$$

Notation: Here it's important to understand the distinction between $\frac{\partial}{\partial x}$ and $\frac{d}{dx}$. Note that L is a function of three variables which we denote x, y, and $\dot{y}$. As usual, $\frac{\partial L}{\partial x}$, $\frac{\partial L}{\partial y}$, and $\frac{\partial L}{\partial \dot{y}}$ denote its partials with respect to those variables. The composed function $L(x, y(x), y'(x))$ is a function of one variable (namely x); its derivative is written $\frac{dL}{dx}$. Thus (*) can be written as

$$D_2 L(x, y(x), y'(x)) - \frac{d}{dx} D_3 L(x, y(x), y'(x)).$$

Proof. Consider a C^2 function $u : [x_0, x_1] \to \mathbf{R}$ that vanishes on the endpoints: $u(x_0) = u(x_1) = 0$. Then the function $y(\cdot) + u(\cdot)$ also satisfies the boundary conditions, so

$$\mathcal{L}[(y(\cdot)] \leq \mathcal{L}[y(\cdot) + u(\cdot)].$$

More generally,

$$\mathcal{L}[y(\cdot)] \leq \mathcal{L}[y(\cdot) + su(\cdot)]$$

for every $s \in \mathbf{R}$. Thus the function $f(s) := \mathcal{L}[y(\cdot)] \leq \mathcal{L}[y(\cdot)+su(\cdot)]$ has its minimum at 0, so $f'(0) = 0$ if the derivative exists.

In fact, the derivative does exist and we can calculate it as follows:

$$\begin{aligned} f'(s) &= \frac{d}{ds}\int_{x_0}^{x_1} L(x, y(x) + su(x), y'(x) + su'(x))\, dx \\ &= \int_{x_0}^{x_1} \frac{d}{ds} L(x, y(x) + su(x), y'(x) + su'(x))\, dx \\ &= \int_{x_0}^{x_1} \left(\frac{\partial L}{\partial y}(x, y(x) + su(x), y'(x) + su'(x))u(x) \right. \\ &\qquad \left. + \frac{\partial L}{\partial \dot{y}}(x, y(x) + su(x), y'(x) + su'(x))u'(x) \right) dx \end{aligned}$$

so

$$f'(0) = \int_{x_0}^{x_1} \left(\frac{\partial L}{\partial y}(x, y(x), y'(x))u(x) + \frac{\partial L}{\partial \dot{y}}(x, y(x), y'(x))u'(x) \right) dx$$

or simply

$$f'(0) = \int_{x_0}^{x_1} \left(\frac{\partial L}{\partial y} u + \frac{\partial L}{\partial \dot{y}} \frac{du}{dx} \right) dx$$

Integrating the second expression by parts gives

$$f'(0) = \int_{x_0}^{x_1} \left(\frac{\partial L}{\partial y} u - \frac{d}{dx}\left(\frac{\partial L}{\partial \dot{y}} \right) u \right) dx + \left(\frac{\partial L}{\partial \dot{y}} u \right)\Big|_{x_0}^{x_1}.$$

The last expression vanishes since $u(x_0) = u(x_1) = 0$. Thus

$$f'(0) = \int_{x_0}^{x_1} \left(\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial \dot{y}} \right) \right) u\, dx$$

Thus we have shown

$$\int_{x_0}^{x_1} \left(\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial \dot{y}} \right) \right) u\, dx = 0.$$

This must hold for all C^2 functions that vanish on the boundary.

Hence

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial \dot{y}} \right) = 0. \tag{†}$$

□

If the last step of the proof is not clear, suppose that $\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial \dot{y}} \right)$ were nonzero at some point. Then it would be nonzero on some open interval $(a, b) \subset [x_0, x_1]$.

Indeed, it would be everywhere > 0 or everywhere < 0 on that interval. Now let u be a C^2 function that is > 0 on (a, b) and 0 on $\mathbf{R} \setminus (a, b)$. For example, we could let

$$u(x) = \begin{cases} (x-a)^4(b-x)^4 & \text{if } x \in [a,b], \text{ and} \\ 0 & \text{if } x \notin [a,b]. \end{cases}$$

Then the integral in (†) is nonzero, a contradiction.

Note that $y(\cdot)$ being a solution of the Euler-Lagrange equation does **not** imply that $y(\cdot)$ minimizes $\mathcal{L}$. Rather, it means that $y(\cdot)$ passes the first derivative test for being a minimum. However, as in calculus, if we're lucky, then the first derivative will narrow our search down to a few possibilities.

1. Example: Shortest Curve

Let's try to find a function that minimizes the arclength of its graph

$$\mathcal{L}[y(\cdot)] = \int_{x_0}^{x_1} \sqrt{1 + \dot{y}^2}\, dx.$$

Here $L(x, y, \dot{y}) = \sqrt{1 + \dot{y}^2}$. Thus

$$\frac{\partial L}{\partial y} = 0 \quad \text{and} \quad \frac{\partial L}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{1+\dot{y}^2}},$$

so the Euler-Lagrange Equation becomes

$$\begin{aligned} 0 &= \frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial \dot{y}}\right) \\ &= 0 - \frac{d}{dx}\left(\frac{\dot{y}}{\sqrt{1+\dot{y}^2}}\right) \\ &= -\frac{d}{dx}\left(\frac{\dot{y}}{\sqrt{1+\dot{y}^2}}\right). \end{aligned}$$

Thus $\dot{y}/\sqrt{1+\dot{y}^2}$ must be constant, and therefore $\dot{y}$ must be constant. Thus $y = ax + b$ for constants a and b.

From the boundary conditions, we see that

$$y = \frac{y_1 - y_0}{x_1 - x_0}\, x + y_0. \tag{1}$$

We have not proved that the minimum exists. However, we have proved that if the minimum does exist, it must be the function (1).

2. Example: Catenoids

The **Plateau Problem** is the following: given one or more closed curves in $\mathbf{R}^3$, find a surface of least possible area among all surfaces having those curves as boundary. Let us consider a special case of the Plateau Problem: we look for a least area surface whose boundary is a pair of circles, assuming that the minimum exists and is a surface of revolution.

In other words, suppose $0 < x_0 < x_1$ and that $y(\cdot) : [x_0, x_1] \to \mathbf{R}$ is a C^2 function. We can rotate the graph of $y(\cdot)$ about the y-axis to get a surface S of revolution in $\mathbf{R}^3$. The area of S is given by

$$\int_{x_0}^{x_1} 2\pi x\sqrt{1+(y')^2}\,dx$$

Let's try to find a function $y(\cdot)$ that minimizes this area (subject to specified boundary conditions $y(x_0) = y_0$ and $y(x_1) = y_1$.)

The problem is equivalent to minimizing

$$\mathcal{L}[y(\cdot)] = \int_{x_0}^{x_1} x\sqrt{1+(y')^2}\,dx$$

Here $L(x, y, \dot{y}) = x\sqrt{1+(\dot{y})^2}$, so

$$\frac{\partial L}{\partial y} = 0, \qquad \frac{\partial L}{\partial \dot{y}} = x\,\frac{\dot{y}}{1+(\dot{y})^2},$$

so the Euler-Lagrange Equation becomes

$$\begin{aligned} 0 &= \frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial \dot{y}}\right) \\ &= 0 - \frac{d}{dx}\left(\frac{x\dot{y}}{\sqrt{1+\dot{y}^2}}\right) \\ &= -\frac{d}{dx}\left(\frac{x\dot{y}}{\sqrt{1+\dot{y}^2}}\right). \end{aligned}$$

Thus

$$\frac{x\dot{y}}{\sqrt{1+\dot{y}^2}} = c.$$

Solving for $\dot{y}$,

$$(x^2 - c^2)(\dot{y})^2 = c^2,$$

or

$$\frac{dy}{dx} = \frac{c}{\sqrt{x^2 - c^2}},$$

so

$$y = \int \frac{c}{\sqrt{x^2 - c^2}}\,dx.$$

To integrate, let $x = c\cosh u$. Then $dx = c\sinh u$ and $x^2 - c^2 = c^2(\cosh^2 u - 1) = c^2 \sinh^2 u$. Thus

$$y = \int c\,du = cu + \hat{c},$$

so

$$u = \frac{y - \hat{c}}{c}.$$

Taking cosh of both sides gives

$$\frac{x}{c} = \cosh\left(\frac{y - \hat{c}}{c}\right). \tag{2}$$

One can think of $x = \cosh y$ as the "basic" solution. All other solutions come be dilating the fundamental solution (by $(x, y) \mapsto (cx, cy)$) and then translating in the y-direction (by $(x, y) \mapsto (x, y + \frac{\hat{c}}{c})$.)

Of course we try to choose c and $\tilde{c}$ so that the solution curve passes through the point (x_0, y_0) and (x_1, y_1).

Definition 2. The surface of revolution given by $\sqrt{x^2 + z^2} = \cosh y$ is called a **catenoid**. More generally, if we apply a translation, rotation, and dilation to that surface, the resulting surface is also called a catenoid.

Suppose $(x_0, y_0) = (1, -h)$ and $(x_1, y_1) = (0, h)$. (Geometrically, this means that in $\mathbf{R}^3$, the boundary of our surface consists of two circles of radius 1, one in the plane the $x = -h$ and the other in the plane $x = h$.)

Exercise 1. Show that if h is small, then there are exactly two curves of the form (2) that pass through the points $(1, -h)$ and $(1, h)$. Which one has less area?

Exercise 2. Show that if h is large, then there is **no** curve of the form (2) passing through $(1, -h)$ and $(1, h)$. What can you conclude?

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A curious thing happened in the catenoid example above. We were looking for a function $y = y(x)$, but we ended up with a function $x = x(y)$. Suppose for example that $(x_0, y_0) = (\cosh 1, -1)$ and $(x_1, y_1) = (\cosh 2, 2)$. Then

$$x = \cosh y, \quad -1 \leq y \leq 2$$

is a curve that has the specified endpoints. However, it cannot be written in the form $y = y(x)$.

So is our analysis valid? It is in the following sense. Suppose we allow curves C joining (x_0, y_0) and (x_1, y_1) that are not necessarily graphs. Then Theorem 1 does apply to each portion that is a graph.

Conserved Quantities

Note that if $L(x, \dot{y})$ is actually a function of x and $\dot{y}$ alone, then $\frac{\partial L}{\partial y} = 0$, so the Euler-Lagrange equation simplifies to

$$\frac{d}{dx}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0.$$

Thus $y(\cdot)$ is a solution if and only if

$$\frac{\partial L}{\partial \dot{y}} = c$$

for some constant c. Thus $\frac{\partial L}{\partial \dot{y}}$ is a "conserved quantity"; it doesn't change as x changes.

Similarly, if $L = L(y, \dot{y})$ a a function y and $\dot{y}$ alone, there is also a conserved quantity:

Theorem 3. *Suppose $L = L(y, \dot{y})$. Then a nonconstant function $y(\cdot)$ is solution of the Euler-Lagrange equation if and only if the quantity*

$$Q = \dot{y}\,\frac{\partial L}{\partial \dot{y}} - L$$

is constant (i.e., independent of x).

Proof. Note that

$$\frac{dQ}{dx} = \ddot{y}\frac{\partial L}{\partial \dot{y}} + \dot{y}\frac{d}{dx}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{dL}{dx}$$

By the chain rule,

$$\frac{dL}{dx} = \frac{\partial L}{\partial y}\dot{y} + \frac{\partial L}{\partial \dot{y}}\,\ddot{y}.$$

(Note that if $L = L(x, y, \dot{y})$, the right hand side would also include $\frac{\partial L}{\partial x}$.) Thus

$$\frac{dQ}{dx} = \dot{y}\left(\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial \dot{y}}\right)\right).$$

□

Vector-Valued Functions

The derivation of the Euler-Lagrange Equation works equally for vector-valued function $y : [x_0, x_1] \to \mathbf{R}^n$. Here L will be a function of $2n + 1$ variables:

$$L = L(x, y_1, \ldots, y_n, \dot{y}_1, \ldots, \dot{y}_n).$$

In this case, the Euler-Lagrange Equation becomes a system of differential equations:

$$\frac{\partial L}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial L}{\partial \dot{y}_i}\right) = 0 \quad (1 \le i \le n).$$