Math 63CM discussion section problems for April 12, 2019.

These problems are *not to be turned in*, but are provided in the hope that you will find some of them interesting and instructive. Feel free to come to office hours if you want to discuss any of them beyond what we have time for in section.

1. *The condition in Peano's existence theorem is necessary*. Consider the ODE

$$\dot{x} = \begin{cases} 1 & x \le 0 \\ a & x > 0 \end{cases}$$
$$x(0) = 0.$$

Does this ODE have a solution?

2. *Matrix exponentials*. Find the exponentials of $t$ times the following matrices:

   (a) $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$.

   (b) $\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}$.

   (c) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

   (d) $\begin{pmatrix} 1 & 2 \\ 3 & -3 \end{pmatrix}$.

   (e) $\begin{pmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{pmatrix}$.

   (f) $\begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}$.

3. Consider the *Lotka–Volterra equations*

$$\dot{x} = \alpha x - \beta x y$$
$$\dot{y} = \delta x y - \gamma y,$$

where $\alpha, \beta, \gamma, \delta > 0$ and $x(0), y(0) > 0$. These equations model the interaction of a predator and a prey species.

   (a) Is the corresponding vector field locally Lipschitz?

   (b) Is the corresponding vector field globally Lipschitz?

   (c) What are the equilibrium points of this model?

   (d) Must solutions exist and stay positive for all time?

      i. Hint for one method: draw the vector field and argue geometrically.

      ii. Hint for another method: consider the quantity $F(x,y) = \delta x - \gamma \log x + \beta y - \alpha \log y$.

4. Let $U \subset \mathbf{R}^n$ be open and path-connected and let $x, y \in U$. Show that there exists a $C^1$ homeomorphism $\Psi : U \to U$ so that $\Psi(x) = y$.