Math 63CM discussion section problems for April 19, 2019.

These problems are *not to be turned in*, but are provided in the hope that you will find some of them interesting and instructive. Feel free to come to office hours if you want to discuss any of them beyond what we have time for in section.

1. Solve the ODE $x''(t) + 6x'(t) + 9x(t) = 1$, $x(0) = 1$, $x'(0) = 1$, using matrix exponentials.

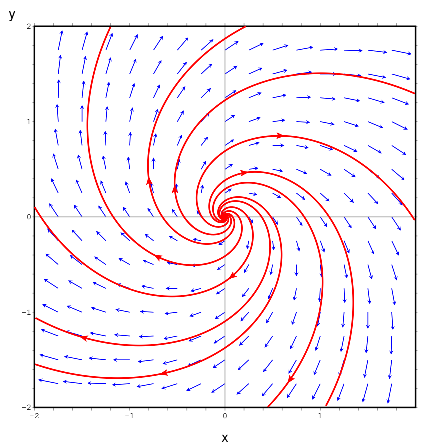
2. If we are interested in the linear ODE $\dot{\mathbf{x}} = A\mathbf{x}$, then we have learned that certain properties of the eigenvalues of A are the key to understanding the behavior. Suppose that A is a 2×2 matrix, but I just tell you $\operatorname{tr} A$ and $\det A$ (not the entries of A). How can you figure out the eigenvalues of A? How can you characterize the behavior of the ODE?

3. Each figure below shows the vector field and several integral curves for one of the ODEs

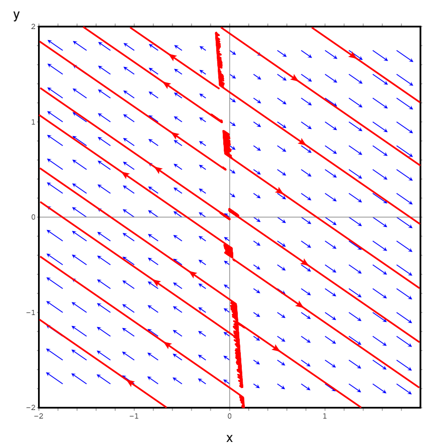
$$\text{(i)}\ \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}-2 & -1\\ -1 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix},\quad \text{(ii)}\ \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}-15.2 & -9.6\\ -9.6 & -0.8\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}\quad \text{(iii)}\ \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}2.6 & 0.8\\ 0.8 & 1.4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}\quad \text{(iv)}\ \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}13 & 1\\ -9 & 7\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$\text{(v)}\ \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}2 & 3\\ -3 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix},\qquad \text{(vi)}\ \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}2 & 26\\ -1/2 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}\qquad \text{(vii)}\ \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}-3 & 3\\ -3 & -3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

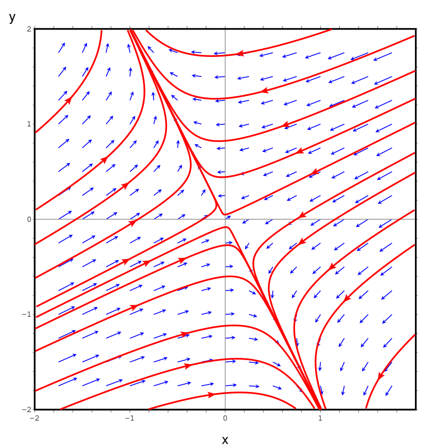
Which is which?



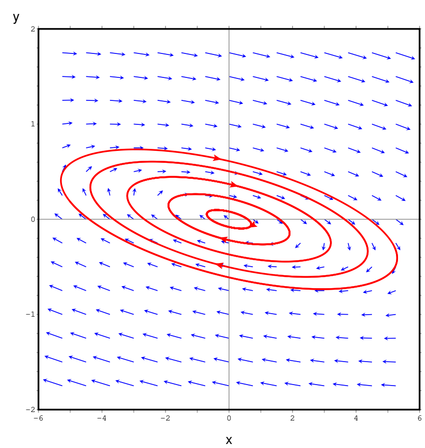
(a)



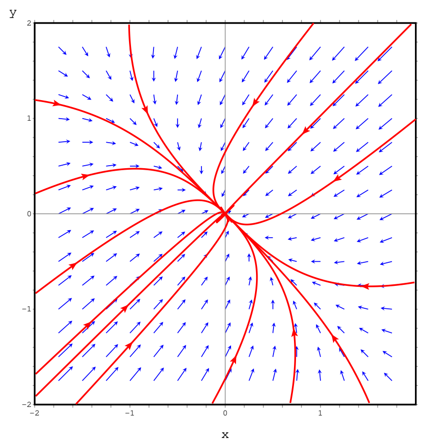
(b)



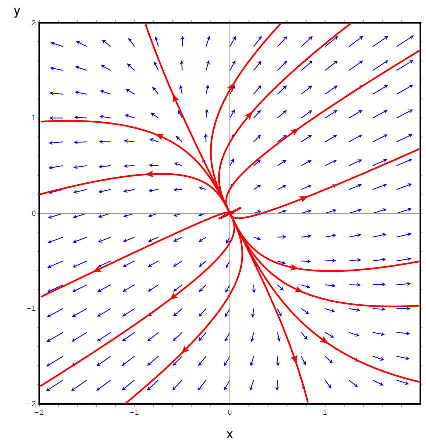
(c)



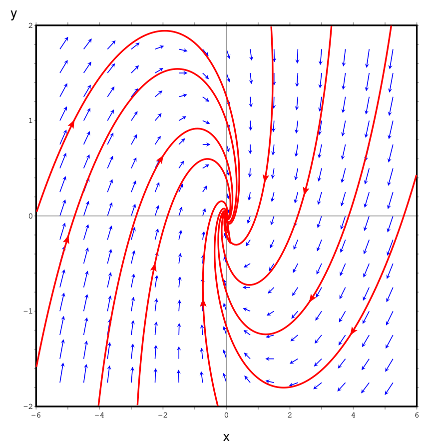
(d)



(e)



(f)



(g)

4. Write the following matrices as the sum of a diagonalizable matrix and a nilpotent matrix.

$$\text{(a)} \begin{pmatrix} 2 & 2 & 3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{pmatrix} \qquad \text{(b)} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

5. *Imaginary eigenvalues.*

   (a) Suppose that A is a diagonalizable $n \times n$ matrix, all of whose eigenvalues are pure imaginary. Show that, for any $x \in \mathbf{R}^n$, $\{e^{tA} \mid t \in \mathbf{R}\}$ is a bounded set. Must $\{e^{tA} \mid t \in \mathbf{R}\}$ be closed?

   (b) Show that if A is a 2×2 real matrix with nonzero pure imaginary eigenvalues, then A is diagonalizable.

   (c) Let $B = \begin{pmatrix} \mathrm{i} & 1 \\ 0 & \mathrm{i} \end{pmatrix}$. Show that $\{e^{tB} \mid t \in \mathbf{R}\}$ is unbounded.

   (d) Let

$$C = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

   Show that $\{e^{tC} \mid t \in \mathbf{R}\}$ is unbounded.

   (e) What do the matrices B and C have to do with each other?