Math 63CM discussion section problems for April 26, 2019.

These problems are *not to be turned in*, but are provided in the hope that you will find some of them interesting and instructive. Feel free to come to office hours if you want to discuss any of them beyond what we have time for in section.

1. For the following systems of ODEs....
   (a) Find all equilibrium points of the system.
   (b) Linearize the system about the equilibrium points. Sketch the phase portrait of the linearized system. Analyze the behavior of the linearized system.
   (c) To what extent do you expect the behavior of the linearized system to reflect that of the nonlinear system?
   (d) Solve the nonlinear system explicitly. Sketch the phase portrait and compare it to that of the linearized system.
      i.
      $$\dot{x} = x + y^2$$
      $$\dot{y} = -y.$$
      (*Hint for part (d):* you do this explicitly, but you can also do it easily by changing coordinates by $u = x + \frac{1}{3}y^2$, $v = y$.) For this problem, what are the stable and unstable manifolds?
      ii.
      $$\dot{x} = \frac{1}{2}x - y - \frac{1}{2}(x^3 + y^2 x)$$
      $$\dot{y} = x + \frac{1}{2}y - \frac{1}{2}(y^3 + x^2 y).$$
      (*Hint for part (d):* use polar coordinates.)
      iii.
      $$\dot{x} = -y + \gamma x(x^2 + y^2)$$
      $$\dot{y} = x + \gamma y(x^2 + y^2),$$
      where $\gamma \in \mathbf{R}$ is a parameter. Consider separately the cases $\gamma < 0$, $\gamma = 0$, and $\gamma > 0$. (*Hint for part (d):* use polar coordinates.)
      iv.
      $$\dot{x} = x^2$$
      $$\dot{y} = -y.$$
2. Find a global change of coordinates to turn the system
   $$\dot{x} = x + y^2$$
   $$\dot{y} = -y$$
   $$\dot{z} = -z + y^2$$
   into a linear one. What are the stable and unstable manifolds?
3. Consider the system of ODEs (in polar coordinates)
   $$\dot{r} = r - r^2$$
   $$\dot{\theta} = \sin\theta + a$$
   where $a$ is some parameter. How does the qualitative behavior of the system depend on the parameter?
4. Consider a pendulum that is allowed to rotate 360° around its center of rotation. Write down an ODE governing its movement. (It should be an ODE in the *phase* space $(\theta, \dot{\theta})$.) What are the equilibrium points? Are they stable or unstable? (Your mathematical results should match your intuition!)