Math 63CM discussion section problems for May 3, 2019.

These problems are *not to be turned in*, but are provided in the hope that you will find some of them interesting and instructive. Feel free to come to office hours if you want to discuss any of them beyond what we have time for in section.

1. Consider the system of differential equations

$$\begin{aligned}\dot{x} &= (\varepsilon x + 2y)(z+1)\\ \dot{y} &= (-x + \varepsilon y)(z+1)\\ \dot{z} &= -z^3.\end{aligned}$$

   (a) Find all equilibrium points for this system.

   (b) Linearize the system around the equilibrium points. What does the linearization tell you?

   (c) Find a Lyapunov function for this ODE. *Hint*: consider functions of the form $L(x,y,z) = ax^2 + by^2 + cz^2$. What can you conclude from the Lyapunov function?

2. Consider the equation of a pendulum, with frictional force proportional to velocity:

$$\begin{aligned}\dot{\theta} &= v\\ \dot{v} &= -bv - \sin\theta.\end{aligned}$$

   (Here $b \geq 0$ is the coefficient of friction; all of the other constants have been rescaled so that the coefficients are 1.)

   (a) Why is this the equation of a damped pendulum? On what domain should θ and v be defined?

   (b) What are the equilibrium points of the system? Does that match your physical intuition?

   (c) Linearize the system around the equilibrium points. What does the linearization tell you?

   (d) Consider the function

$$E = \frac{1}{2}v^2 + 1 - \cos\theta.$$

      i. Show that E is a Lyapunov function for the system.

      ii. What is the physical interpretation of E?

      iii. If $b > 0$, what can you conclude from the Lyapunov function?

      iv. If $b = 0$, use the Lyapunov function to sketch the phase portrait of the system.

3. Consider the ODE

$$\begin{aligned}\dot{x} &= -x^3\\ \dot{y} &= -y(x^2+z^2+1)\\ \dot{z} &= -\sin z.\end{aligned}$$

   (a) Find all equilibrium points of the system.

   (b) Linearize the system around the origin. What does the linearization tell you?

   (c) Show that $F(x,y,z) = x^2 + y^2 + z^2$ is a Lyapunov function for this system. (On what set is it a Lyapunov function?) What can you conclude from this?