Math 63CM discussion section problems for May 10, 2019.

These problems are *not to be turned in*, but are provided in the hope that you will find some of them interesting and instructive. Feel free to come to office hours if you want to discuss any of them beyond what we have time for in section.

1. Suppose that V is smooth and that $\dot{x} = -\nabla V(x)$.

   (a) Suppose that there is a $y \in \mathbf{R}^n$ and a sequence of times $t_n \to \infty$ so that $\lim_{n\to\infty} |x(t_n) - y| = 0$. Prove that y is an equilibrium point.

   (b) Show that all eigenvalues of the linearized system at any equilibrium point are real.

2. Compute
$$I(\xi) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\xi x} \mathrm{e}^{-\pi x^2} \, \mathrm{d}x.$$
   Hint: find a differential equation solved by $I(\xi)$.

3. Suppose I give you an $n \times n$ complex matrix $A \in \mathbf{C}^{n\times n}$. Consider a $2n \times 2n$ real matrix $\phi(A) \in \mathbf{R}^{(2n)\times(2n)}$ by replacing each element $a + \mathrm{i}b$ of A with the 2×2 matrix
$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

   (a) Show that $\phi(A)\phi(B) = \phi(AB)$.

   (b) How are the eigenvalues and eigenvectors of $\phi(A)$ related to those of A?

4. Suppose that x and y are points on $S^2 = \{(x_1, x_2, x_3) \in \mathbf{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$. Find, with proof, the shortest curve in S^2 connecting x and y. (Derive Euler–Lagrange equations.)

5. Consider a damped pendulum with constant forcing, given by the equations, with $k \geq 0$.
$$\begin{aligned} \dot{\theta} &= v \\ \dot{v} &= -bv - \sin\theta + k. \end{aligned}$$

   (a) Find the equilibria of the system and determine if they are stable.

   (b) Draw the b–k parameter plane and label the regions with different numbers of equilibrium points.

   (c) Show that if $k > 1$, there is a unique periodic solution to the system. *Hint:* use the energy function $E = \frac{1}{2}v^2 - \cos\theta + 1$ and the fact that, along a periodic solution, the change in energy will be constant.

   (d) Show that there are parameter values that admit both a stable equilibrium and a periodic solution.