Math 63CM discussion section problems for May 17, 2019.

These problems are *not to be turned in*, but are provided in the hope that you will find some of them interesting and instructive. Feel free to come to office hours if you want to discuss any of them beyond what we have time for in section.

1. Find the α- and ω-limit sets of all trajectories of the following systems:

   (a) $r' = r - r^2$, $\theta' = 1$.

   (b) $r' = r^3 - 3r^2 + 2r$, $\theta' = 1$.

   (c) $r' = \sin r$, $\theta' = 1$.

   (d) $x' = \sin x \sin y$, $y' = -\cos x \cos y$.

2. Suppose that $x(t)$ is a bounded trajectory. Show that the ω-limit set of x is connected.

3. Consider the three-dimensional system given by (in spherical coordinates)

$$\begin{aligned} r' &= r(1-r) \\ \theta' &= 1 \\ z' &= \gamma z. \end{aligned}$$

   For which values of γ can you show that there is a neighborhood U of the circle $C = \{z = 0, r = 1\}$ so that for every $(r, \theta, z) \in U$, C is the the ω-limit set of the trajectory started at (r, θ, z)?

4. Consider the system

$$\begin{aligned} x' &= \sin x(-0.1 \cos x - \cos y) \\ y' &= \sin y(\cos x - 0.1 \cos y). \end{aligned}$$

   Show that this system has a source at $x = y = \pi/2$. Show that all solutions emanating from the source have ω-limit sets equal to the square bounded by $x = 0, \pi$ and $y = 0, \pi$.

5. Let A be an annular region in $\mathbf{R}^2$ with smooth inner and outer boundaries. Suppose that F is a vector field on A so that F has no equilibrium points and F points into the interior of the annulus along the boundary of A. Consider the ODE $\dot{x} = F(x)$ in A.

   (a) Show that there is a closed orbit in A.

   (b) Suppose that there are only finitely many closed orbits of A. Show that any closed orbit of A must encircle the central region of A.

      i. *Challenge:* Is this true if there are infinitely many closed orbits?

   (c) Show that if there are only finitely many closed orbits in A, then there is a periodic solution which has trajectories of $\dot{x} = F(x)$ spiraling towards it from both sides.

6. Consider the system

$$\begin{aligned} x' &= -y - \left(\frac{x^4}{4} - \frac{x^2}{2} + \frac{y^2}{2}\right)(x^3 - x) \\ y' &= x^3 - x - \left(\frac{x^4}{4} - \frac{x^2}{2} + \frac{y^2}{2}\right) y. \end{aligned}$$

   (a) Find all equlibrium points.

   (b) Determine the types of these equilibria.

   (c) Prove that all nonequilibrium solutions have ω-limit sets consisting of either one or two homoclinic orbits (orbits that connect two equilibria) plus a saddle point.