

Math 63CM discussion section problems for May 24, 2019.

These problems are *not to be turned in*, but are provided in the hope that you will find some of them interesting and instructive. Feel free to come to office hours if you want to discuss any of them beyond what we have time for in section.

The *Fisher–Kolmogorov–Petrovskii–Piskunov* (*Fisher–KPP*, or *FKPP*, or *KPP*) equation is a partial differential equation given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u),$$

where $u = u(t, x)$, $t, x \in \mathbf{R}$. The function u represents the density of some kind of animal in time and space. The second spatial derivative of u represents the spreading of the animal in space. We should have $u(t, x) \in [0, 1]$ to be biologically relevant, since the density of animals should be positive and 1 represents the carrying capacity of the environment.

1. Let $c > 0$. Show that if $v(t, x) = u(t, x + ct)$, then v satisfies the partial differential equation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + c \frac{\partial v}{\partial x} + v(1-v). \quad (1)$$

2. If there is a function $v_0 = v_0(x)$ so that $v(t, x) = v_0(x)$ for all t and v solves (1), what does this mean about u ?

For the rest of this worksheet, we will try to find solutions to the ODE

$$0 = v'' + cv' + v(1-v). \quad (2)$$

3. For a particular value of c , there is a solution of (2) of the form

$$v(x) = \frac{1}{(1 + e^{ax})^2}.$$

Find c and a . What does this mean in terms of u ?

4. Write (2) as a first-order ODE in two variables $f = v$, $g = v'$. Find all the equilibrium points of this ODE and classify them by their linearizations. Which solutions do these give in terms of u ?
5. Show that if $c < 2$, then we cannot expect there to be a solution of (2) that stays in $(0, 1)$ for all time (positive and negative)
6. Suppose that $c \geq 2$ and define β so that $\beta + \frac{1}{\beta} = c$. Define

$$D = \{(f, g) \mid 0 < f < 1, -\beta f < g < 0\}.$$

Show that D is a positively-invariant set.

7. Show that D contains no limit cycles.
8. Show that if $c \geq 2$, then there is a solution to (2) that stays in $(0, 1)$ for all time, and moreover that this solution connects 0 and 1. Interpret this in terms of u .