# MATH 63CM HOMEWORK 1

1. Find a smooth function $F : \mathbf{R} \to \mathbf{R}$ with the following property. If $x : (a, b) \to \mathbf{R}$ is a solution of $x'(t) = F(x(t))$, then $a > -\infty$ and $b < \infty$.

2. Let $F : \mathbf{R}^N \to \mathbf{R}^N$ be a locally Lipschitz vectorfield such that

$$\frac{|F(p)|}{1 + |p|} \le C < \infty$$

for all $p$. Prove that for each $x_0 \in \mathbf{R}^N$, there is a solution $x : [0, \infty) \to \mathbf{R}^N$ of the initial value problem

$$\begin{aligned} x'(t) &= F(x(t)), \\ x(0) &= x_0. \end{aligned}$$

3. Suppose $F : U \subset \mathbf{R}^d \mapsto \mathbf{R}^N$ is locally Lipschitz, i.e., that for every $p \in U$, there is an $r > 0$ and $C < \infty$ such that

$$x, y \in U \cap \mathbf{B}(p, r) \implies |F(x) - F(y)| \le C|x - y|.$$

Suppose that $K$ is a compact subset of $U$. Prove that $F|K$ is Lipschitz, i.e., that there is an $L < \infty$ such that

$$x, y \in K \implies |F(x) - F(y)| \le L|x - y|.$$

4. Suppose that $x_i : I \subset \mathbf{R} \to \mathbf{R}^3$ $(i = 1, \dots, n)$ are functions such that $x_i(t) \ne x_j(t)$ for all $i \ne j$ and $t \in I$, and that

$$m_i x_i''(t) = \sum_{j \ne i} F_{ij}(x_i(t), x_j(t))$$

where

$$F_{ij} : \{(p, q) \in \mathbf{R}^3 \times \mathbf{R}^3 : p \ne q\} \to \mathbf{R}^3$$

are smooth functions such that

$$(*) \qquad F_{ij}(p, q) \equiv -F_{ji}(q, p).$$

(a) Prove that the total momentum $\sum_i m_i x_i'(t)$ is constant.
(b) Suppose (in addition to (*)) that $F_{ij}(p, q)$ is a scalar multiple of $p - q$ for all $p$ and $q$. Prove that the total angular momentum

$$\sum m_i x_i(t) \times x_i'(t)$$

is constant.

(c) Suppose that $F_{ij}(p,q) = \phi'_{ij}(|p-q|)\frac{q-p}{|q-p|}$ for some smooth function $\phi_{ij} : (0,\infty) \to \mathbf{R}$ with $\phi_{ij} = \phi_{ji}$. Prove that the total energy

$$\sum_i \left( \frac{1}{2} m_i |x_i'|^2 + \frac{1}{2} \sum_{j \neq i} \phi_{ij}(|x_i - x_j|) \right)$$

(or, equivalently, $\sum_i \frac{1}{2} m_i |x_i'|^2 + \sum_{i<j} \phi_{ij}(|x_i - x_j|)$) is constant.

5. Define $F : \mathbf{R} \to \mathbf{R}$ by

$$F(x) = \begin{cases} -x \log(|x|) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Note that $x(t) \equiv 0$ is a solution of the initial value problem

$$x'(t) = F(x(t)),$$
$$x(0) = 0.$$

Show that it is the unique solution. [Note: $F$ is **not** Locally Lipschitz, so we cannot just apply the standard uniqueness theorem.]

**Hint**: Suppose to the contrary that there is a solution on some interval $I$ containing 0 such that $x(0) = 0$ and $x(T) \neq 0$ for some $T \in I$. Let $t_0$ be the time closest to $T$ such that $x(t_0) = 0$. Now consider the solution $x(\cdot)$ on $(t_0, T]$ (if $t_0 < T$) or on $[T, t_0)$ (IF $t_0 > T$).

6. Brendle problem 1.1.

7. Brendle problem 1.2.

8. Brendle problem 1.3.

9. Brendle problem 1.5