1 Take $F(x) = 1 + x^2$. If $x'(t) = 1 + x(t)^2$, then we can write

$$\frac{\mathrm{d}}{\mathrm{d}t}\arctan x(t) = \frac{x'(t)}{1+x(t)^2} = 1,$$

so

$$\arctan x(t) = t + C$$

for some constant $C \in \mathbf{R}$, so $x(t) = \tan(t+C)$. But note that $\tan(t+C) \to \pm\infty$ as $t \to C \pm \pi/2$, so x can only be defined for $t \in C + (-\pi/2, \pi/2)$.

2 By Picard's theorem and its corollaries, we know that there is a maximal T so that there is a solution $x : [0,T)$ of the initial value problem, and moreover that $\lim_{t\to T^-} |x(t)| = \infty$. On the other hand, we note that by assumption we have

$$|x'(t)| = |F(x(t))| \le C(1 + |x(t)|),$$

and therefore, by the Cauchy–Schwarz inequality, that

$$\left|\frac{\partial}{\partial t}\log(1+|x(t)|^2)\right| = \left|\frac{x'(t)\cdot x(t)}{1+|x(t)|^2}\right| \le \frac{|x'(t)|\cdot|x(t)|}{1+|x(t)|^2} \le \frac{C(1+|x(t)|)|x(t)|}{1+|x(t)|^2} \le CC_1,$$

where

$$C_1 = \max_{a\ge 0}\frac{a^2+a}{a^2+1} < \infty.$$

By the mean value theorem, this means that, for all $t \ge 0$, we have

$$\left|\log(1+|x(t)|^2)\right| \le CC_1 t + \left|\log(1+|x(0)|^2)\right|,$$

which makes it impossible for $x(t)$ to approach infinity as t approaches any finite T.

3 Let $r_p > 0$ and $C_p < \infty$ be the constants so that

$$x, y \in U \cap \mathbf{B}(p, r_p) \implies |F(x) - F(y)| \le C_p|x - y|.$$

Let $Q_p = \mathbf{B}(p,r_p) \times \mathbf{B}(p,r_p) \subset \mathbf{R}^n \times \mathbf{R}^n$ and let $\Delta_K = \{(x,x) \mid x \in K\} \subset \mathbf{R}^n \times \mathbf{R}^n$. Note that

$$\bigcup_{p\in K} Q_p \supset \Delta_K.$$

Since Δ_K is compact, we can find a finite collection $p_1, \ldots, p_N$ so that

$$Q := \bigcup_{j=1}^{N} Q_{p_j} \supset \Delta_K.$$

Define

$$L_1 = \max_{j=1}^{N} C_{p_j} < \infty.$$

Since Δ_K is compact and $(\mathbf{R}^n \times \mathbf{R}^n) \setminus Q$ is closed, and these two sets are disjoint, we have that

$$R := \operatorname{dist}(\Delta_K, (\mathbf{R}^n \times \mathbf{R}^n) \setminus Q) > 0.$$

Now if $x, y \in K$ are such that $|x-y| < R$, then $\operatorname{dist}((x,y),\Delta_K) < \operatorname{dist}((x,y),(x,x)) = |x-y| < R$, so $(x,y) \in Q$. By the definition of Q, there exists a $j \in \{1,\ldots,N\}$ so that $(x,y) \in Q_{p_j} = \mathbf{B}(p_j, r_{p_j}) \times \mathbf{B}(p_j, r_{p_j})$, or in other words that $x, y \in \mathbf{B}(p_j, r_{p_j})$, so

$$|f(x) - f(y)| \le C_{p_j}|x-y| \le L_1|x-y|. \tag{1}$$

.

Let $L_2 = \frac{2}{R}\max_{x\in K}|f(x)|$. If $x, y \in K$ are such that $|x - y| \ge R$, then we have that

$$|f(x) - f(y)| \le |f(x)| + |f(y)| \le RL_2 \le L_2|x-y|. \tag{2}$$

Combining (1) and (2), we see that if $L = \max\{L_1, L_2\}$, then for any $x, y \in K$ we have $|f(x) - f(y)| \le L|x - y|$, which was the goal.

4

a We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{n} m_i x_i' = \sum_{i=1}^{n} m_i x_i'' = \sum_{i=1}^{n}\sum_{j\neq i} F_{ij}(x_i,x_j) = \sum_{\substack{i,j\in\{1,\dots,n\}\\ i<j}} [F_{ij}(x_i,x_j)+F_{ji}(x_j,x_i)] = 0$$

by (*).

b We write the assumption as $F_{ij}(p,q) = c_{i,j}(p,q)(p-q)$ for some $c_{i,j}(p,q)\in\mathbf{R}$. The condition (*) then becomes

$$c_{i,j}(p,q) = c_{j,i}(q,p). \tag{3}$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{n} m_i x_i \times x_i' = \sum_{i=1}^{n} m_i x_i' \times x_i' + \sum_{i=1}^{n} m_i x_i \times x_i''.$$

The first term is zero since the cross product is antisymmetric. For the second term, we have

$$\begin{aligned}\sum_{i=1}^{n} m_i x_i \times x_i'' &= \sum_{i=1}^{n} x_i \times (m_i x_i'') = \sum_{i=1}^{n} x_i \times \left(\sum_{j\neq i} F_{ij}(x_i,x_j)\right)\\ &= \sum_{i=1}^{n} x_i \times \left(\sum_{j\neq i} c_{i,j}(x_i,x_j)(x_i-x_j)\right) = -\sum_{i=1}^{n}\sum_{j\neq i} c_{i,j}(x_i,x_j) x_i \times x_j\\ &= -\sum_{\substack{i,j\in\{1,\dots,n\}\\ i<j}} \left[c_{i,j}(x_i,x_j)x_i\times x_j + c_{j,i}(x_j,x_i)x_j\times x_i\right] = 0\end{aligned}$$

by the antisymmetry of the cross product and (3).

c We have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\sum_i\left(\frac{1}{2}m_i|x_i'|^2 + \frac{1}{2}\sum_{j\neq i}\phi_{ij}(|x_i-x_j|)\right)\\ &\quad= \sum_i\left(m_i x_i'\cdot x_i'' + \frac{1}{2}\sum_{j\neq i}\phi_{ij}'(|x_i-x_j|)\frac{(x_i-x_j)\cdot(x_i'-x_j')}{|x_i-x_j|}\right)\\ &\quad= \sum_i\left(x_i'\cdot\sum_{j\neq i}F_{ij}(x_i,x_j) + \frac{1}{2}\sum_{j\neq i}\phi_{ij}'(|x_i-x_j|)\frac{(x_i-x_j)\cdot(x_i'-x_j')}{|x_i-x_j|}\right)\\ &\quad= \sum_i\sum_{j\neq i}\left(x_i'\cdot\phi_{ij}'(|x_i-x_j|)\frac{x_j-x_i}{|x_j-x_i|} + \frac{1}{2}\phi_{ij}'(|x_i-x_j|)\frac{(x_i-x_j)\cdot(x_i'-x_j')}{|x_i-x_j|}\right)\\ &\quad= \sum_{\substack{i,j\in\{1,\dots,n\}\\ i<j}}\left(x_i'\cdot\phi_{ij}'(|x_i-x_j|)\frac{x_j-x_i}{|x_j-x_i|} + x_j'\cdot\phi_{ji}'(|x_j-x_i|)\frac{x_j-x_i}{|x_j-x_i|}\right) + \sum_{\substack{i,j\in\{1,\dots,n\}\\ i<j}}\phi_{ij}'(|x_i-x_j|)\frac{(x_i-x_j)\cdot(x_i'-x_j')}{|x_i-x_j|}\\ &\quad= \sum_{\substack{i,j\in\{1,\dots,n\}\\ i<j}}\left(\left(x_i'-x_j'\right)\cdot\phi_{ij}'(|x_i-x_j|)\frac{x_j-x_i}{|x_j-x_i|}\right) + \sum_{\substack{i,j\in\{1,\dots,n\}\\ i<j}}\phi_{ij}'(|x_i-x_j|)\frac{(x_i-x_j)\cdot(x_i'-x_j')}{|x_i-x_j|}\\ &\quad= 0,\end{aligned}$$

so the energy

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_i\left(\frac{1}{2}m_i|x_i'|^2 + \frac{1}{2}\sum_{j\neq i}\phi_{ij}(|x_i-x_j|)\right)$$

is constant.

5 Suppose for the sake of contradiction that there exists a solution $x \in C^1([0,T])$ so that $x(0) = 0$, $x(T) \neq 0$, and $x'(t) = -x(t)\log|x(t)|$ for all $t \in [0,T]$. Let $t_0 = \sup\{x \in [0,T] \mid x = 0\}$. Since x is assumed to be continuous and $x(T) \neq 0$, we see that $t_0 < T$. But also, by the continuity of x, we must have

$$x(t_0) = 0. \tag{4}$$

Note that for all $t \in (t_0,T]$ we have $x(t) \neq 0$. Define $y \in C^1((t_0,T])$ by $y(t) = \log|x(t)|$. Then we have

$$y'(t) = \frac{x'(t)}{x(t)} = -\frac{x(t)\log|x(t)|}{x(t)} = -\log|x(t)| = -y(t). \tag{5}$$

Therefore, $y(t) = y(T)e^{T-t}$, since of course this solves the differential equation (5), which has unique solutions since the map $F(x) = -x$ is Lipschitz. Therefore, we have that

$$\lim_{t \downarrow t_0} y(t) = y(T)e^{T-t_0},$$

which implies that

$$|x(t_0)| = \lim_{t\downarrow t_0}|x(t)| = \lim_{t\downarrow t_0}\exp\{|y(t)|\} = \exp\left\{\lim_{t\downarrow t_0}|y(t)|\right\} = \exp\left\{|y(T)|e^{T-t_0}\right\} > 0,$$

contradicting (4).

6 We have

$$\frac{d}{dt}(x(t)(1+t^2)) = (1+t^2)x'(t) + 2tx(t) = (1+t^2)\left[-\frac{2t}{1+t^2}x(t)+1\right] + 2tx(t) = 1+t^2$$

and

$$x(0)(1+0^2) = x(0) = 1.$$

Therefore, we have

$$x(t)(1+t^2) = 1 + \int_0^t (1+s^2)\,ds = 1 + t + \frac{t^3}{3},$$

so

$$x(t) = \frac{1+t+t^3/3}{1+t^2}.$$

Indeed, we can check that

$$x'(t) = \frac{(1+t^2)^2 - 2t(1+t+t^3/3)}{(1+t^2)^2} = 1 - \frac{2t}{1+t^2}x(t)$$

and

$$x(0) = 1,$$

as required.

7 We have

$$\begin{aligned}\frac{d}{dt}\left((t+1)e^{-t}x(t)\right) &= e^{-t}x(t) - (t+1)e^{-t}x(t) + (t+1)e^{-t}x'(t) \\ &= -te^{-t}x(t) + (t+1)e^{-t}\left[\frac{t}{t+1}x(t)+1\right] \\ &= (t+1)e^{-t}\end{aligned}$$

and

$$(0+1)e^{-0}x(0) = 0.$$

Therefore, we have

$$(t+1)e^{-t}x(t) = \int_0^t (s+1)e^{-s}\,ds = -(t+1)e^{-t} + (0+1)e^{-0} + \int_0^t e^{-s}\,ds = -(t+1)e^{-t} + 1 + (1-e^{-t}) = 2-(t+2)e^{-t},$$

so

$$x(t) = \frac{2-(t+2)\mathrm{e}^{-t}}{(t+1)\mathrm{e}^{-t}} = \frac{2}{t+1}\mathrm{e}^{t} - \frac{t+2}{t+1}.$$

Indeed, we can check that

$$x'(t) = -\frac{2}{(t+1)^2}\mathrm{e}^{t} + \frac{2}{t+1}\mathrm{e}^{t} - \frac{t+1-(t+2)}{(t+1)^2} = \frac{2t\mathrm{e}^{t}+1}{(t+1)^2},$$

while

$$\begin{aligned}\frac{t}{t+1}x(t)+1 &= \frac{t}{t+1}\left(\frac{2}{t+1}\mathrm{e}^{t} - \frac{t+2}{t+1}\right)+1\\ &= \frac{2t\mathrm{e}^{t}}{(t+1)^2} + \frac{(t+1)^2-t(t+2)}{(t+1)^2}\\ &= \frac{2t\mathrm{e}^{t}}{(t+1)^2} + \frac{t^2+2t+1-t^2-2t}{(t+1)^2}\\ &= \frac{2t\mathrm{e}^{t}+1}{(t+1)^2}\\ &= x'(t),\end{aligned}$$

as required. Moreover, $x(0) = 0$ as required.

8 We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\left(\frac{x(t)}{1-x(t)}\right) = \frac{x'(t)}{x(t)} + \frac{x'(t)}{1-x(t)} = \frac{x'(t)}{x(t)(1-x(t))} = 1.$$

Therefore, we have a constant C so that

$$\log\left(\frac{x(t)}{1-x(t)}\right) = t + C,$$

so

$$\mathrm{e}^{t+C} = \frac{x(t)}{1-x(t)} = \frac{1}{1-1/x(t)},$$

so

$$1/x(t) = 1 - \mathrm{e}^{-t-C},$$

so

$$x(t) = \frac{1}{1-\mathrm{e}^{-t-C}}.$$

9

a We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\log(1+\sin x(t)) - \log(1-\sin x(t))) &= \frac{x'(t)\cos x(t)}{1+\sin x(t)} + \frac{x'(t)\cos x(t)}{1-\sin x(t)}\\ &= (x'(t)\cos x(t))\left(\frac{2}{1-\sin^2 x(t)}\right)\\ &= 2\frac{x'(t)}{\cos x(t)}\\ &= 2,\end{aligned}$$

so by the fundamental theorem of calculus and the given initial condition,

$$\log(1+\sin x(t)) - \log(1-\sin x(t)) = 2t + \log(1+\sin x(0)) - \log(1-\sin x(0)) = 2t.$$

b We have

$$2t = \log \frac{1+\sin x(t)}{1-\sin x(t)},$$

so

$$1+\sin x(t) = e^{2t}(1-\sin x(t)) = e^{2t} - e^{2t}\sin x(t),$$

so

$$\left(e^{2t}+1\right)\sin x(t) = e^{2t}-1,$$

so

$$\sin x(t) = \frac{e^{2t}-1}{e^{2t}+1} = \frac{e^{t}-e^{-t}}{e^{t}+e^{-t}}, \tag{6}$$

so

$$x(t) = \arcsin\left(\frac{e^{t}-e^{-t}}{e^{t}+e^{-t}}\right).$$

The choice of branch of arcsin comes from the fact that $x(0)=0$ and x must be continuous.

On the other hand, we recall by the addition formula for tangent that

$$\tan(\arctan(e^{t}) - \arctan(e^{-t})) = \frac{e^{t}-e^{-t}}{2},$$

so

$$\arctan(e^{t}) - \arctan(e^{-t}) = \arctan\left(\frac{e^{t}-e^{-t}}{2}\right).$$

Note that by (6), we have

$$\tan x(t) = \frac{\frac{e^{t}-e^{-t}}{e^{t}+e^{-t}}}{\sqrt{1-\left(\frac{e^{t}-e^{-t}}{e^{t}+e^{-t}}\right)^2}} = \frac{e^{t}-e^{-t}}{\sqrt{(e^{t}+e^{-t})^2-(e^{t}-e^{-t})^2}} = \frac{e^{t}-e^{-t}}{\sqrt{e^{2t}+2+e^{-2t}-\left(e^{2t}-2+e^{-2t}\right)}} = \frac{e^{t}-e^{-t}}{2}.$$

Therefore, $\tan x(t) = \tan(\arctan(e^{t}) - \arctan(e^{-t}))$, so $x(t) = \arctan(e^{t}) - \arctan(e^{-t})$ by continuity since the left and right sides agree at 0.