MATH 63CM HOMEWORK 2

DUE 11:59PM ON SUNDAY, APRIL 21

1. (Differential equations with parameters.) Suppose that U is an open subset of $\mathbf{R}^k$, that W is an open subset of $\mathbf{R}^n$, and that $F : U \times W \to \mathbf{R}^n$ is a locally Lipschitz map. For $x \in U$ and $y \in W$, consider the initial value problem

$$\begin{aligned} u'(t) &= F(x, u(t)), \\ u(0) &= y. \end{aligned}$$

Let $I_{x,y}$ be the largest interval for which a solution exists. Denote the solution by $t \in I_{x,y} \mapsto \phi_t(x, y)$. Let $Q = \{(x, y, t) : x \in U,\ y \in W,\ t \in I_{x,y}\}$.

(a). Prove that the map

$$(t, x, y) \in Q \mapsto \phi_t(x, y) \tag{*}$$

is continuous.

(b). If F is C^1, show that the map (*) is C^1.

[Hint for (a) and (b): there is a way to deduce this (with almost no work) from things we proved in class.]

2. Consider an $n \times n$ complex matrix A.

(a). Prove that every eigenvalue of A^*A is real and nonnegative. (Recall that A^* is the matrix whose ij entry is $\overline{a_{ji}}$.)

(b). Show that $\|A\|_{\text{op}}$ is equal to the square root of the largest eigenvalue of A^*A.

3. Suppose $a_0, a_1, \dots$ and z are complex numbers such that the series $\sum_{n=0}^{\infty} a_n z^n$ converges.

(a). Prove that if $0 \le r < |z|$, then $\sum_{n=0}^{\infty} |a_n| r^n < \infty$. [Hint: consider $M = \sup_n |a_n z^n|$.]

(b). Prove that if A is a square, complex matrix with $\|A\|_{\text{op}} < |z|$, then $\sum_{n=0}^{\infty} a_n A^n$ converges. (By definition, this means that the sequence $\sum_{n=0}^{m} a_m A^m$ of partial sums converges.)

4. **(a)**. Suppose that K is a compact subset of $\mathbf{R}^N$ and that $F : K \to \mathbf{R}^N$ is a continuous vectorfield. Suppose that $x_n : [0, T] \to K$ is a sequence of functions such that

$$x_n'(t) = F(x_n(t)) \quad \text{for all } t \in [0,T].$$

Prove that $x_n(\cdot)$ has a subsequence $x_{n(i)}(\cdot)$ that converges uniformly to a limit $x : [0,T] \to K$, and that $x'(t) = F(x(t))$ for $t \in [0,T]$.

(b). Suppose that $F : \mathbf{B}(p,R) \subset \mathbf{R}^n \to \mathbf{R}^n$ is a continuous vectorfield and that $M = \sup |F| < \infty$. Let $\delta < R/(3M)$. Suppose that for each $x \in \overline{\mathbf{B}(p,R/3)}$, there is a **unique** solution $u : [0,\delta] \to \mathbf{B}(p,R)$ of the initial value problem

$$\begin{aligned} u'(t) &= F(u(t)), \\ u(0) &= x. \end{aligned}$$

Denote the solution by $\phi(t,x)$. Show that $(t,x) \in [0,T] \times \overline{\mathbf{B}(p,R/3)} \mapsto \phi(t,x)$ is continuous.

[Hint: if suffices to show that if $(t_i,x_i) \in \overline{\mathbf{B}(p,R/3)} \times [0,T]$ converges to (x,t), then $\phi(t_i,x_i)$ converges to $\phi(t,x)$.]

5. Consider the differential equation:

$$(*) \qquad x'(t) = A'(t)x(t)$$

where $x : [0,T] \mapsto \mathbf{R}^n$, $A(t)$ is an $n \times n$ real matrix, and $t \mapsto A(t)$ is continuous.

(a). Show that if $A(t)$ is antisymmetric for each t and if $x(\cdot)$ is a solution of (*), then $|x(t)|$ is constant.

(b). Show that if $|x(t)|$ is constant (i.e, independent of t) for every solution of (*), then $A(t)$ is antisymmetric for every t.

6. Let D be the differentiation operator, i.e, the operator that takes a differentiable function $t \mapsto u(t)$ to the function $t \mapsto u'(t)$. Thus D^2 is the operator that takes the function $u(\cdot)$ to the function $u''(\cdot)$ (assuming the second derivative exists). Find all solutions of the differential equation

$$u'' - 5u' + 6u = 0.$$

Hint: Rewrite the equation as $D^2u - 5Du + 6u = 0$, or $(D^2 - 5D + 6)u = 0$, or $(D-3)(D-2)u = 0$. Let $w = (D-2)u$.