MATH 63CM HOMEWORK 3

DUE 11:59PM ON SUNDAY, APRIL 28

1. Using the method of factoring differential operators (in problem 6 of hw 2), find the solution of $u'' - 4u' + u = 0$ with the initial values $u(0) = a$ and $u'(0) = b$.

2. Solve the initial value problem in problem **1** by converting the 2nd order differential equation to a first-order vector equation and then using the matrix exponential.

3. Find e^{At} where

$$A = \begin{bmatrix} -1 & 3 \\ 0 & 2 \end{bmatrix}.$$

4. Let A be an $n \times n$ **real** matrix. Suppose A has a purely imaginary eigenvalue, i.e., an eigenvalue of the form ci where c is a nonzero real number (and $i^2 = -1$.) Prove that $x' = Ax$ has a **real**, nonzero, periodic solution. (That, is prove that there is a nonzero solution $x : \mathbf{R} \to \mathbf{R}^n$ such that $x(t+T) \equiv x(T)$ for some $T \neq 0$.)

5. Suppose A is a 4×4 matrix with $\det(\lambda I - A) = (\lambda^2 + 1)^2$. Suppose that A is **not** diagonalizable. Prove that $x'(t) = Ax(t)$ has a solution $x(\cdot)$ such that $|x(t)| \to \infty$ as $t \to \infty$.

6. The **trace** of a square matrix is the sum of the diagonal elements: $\operatorname{tr} A = \sum_i a_{ii}$. **(a).** Prove that if C and D are $n \times n$ matrices, then $\operatorname{tr}(CD) = \operatorname{tr}(DC)$. **(b).** Prove that if A and S are $n \times n$ matrices and if S is invertible, then $\operatorname{tr}(S^{-1}AS) = \operatorname{tr}(A)$. **(c).** Prove that $\det(e^A) = e^{\operatorname{tr} A}$.

7. Problem 2.4 in the text.

8. Suppose that A is an $n \times n$ matrix such that $|\lambda_i| < 1$ for each eigenvalue λ_i, $|\lambda_i| < 1$. Prove that $A^k \to 0$ as $k \to \infty$.