MATH 63CM HOMEWORK 4

DUE 11:59PM ON SUNDAY, MAY 5

1. Suppose that $I \subset \mathbf{R}$ is an open interval and $t \in I \mapsto A(t) \in \mathbf{R}^{n\times n}$ is continuous. Suppose for $j = 1, 2, \ldots, n$ that $\mathbf{x}_j : I \to \mathbf{R}^n$ is a solution of

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t). \tag{*}$$

Suppose also that there is a time $t_0 \in I$ such that $\mathbf{x}_1(t_0), \ldots, \mathbf{x}_n(t_0)$ are linearly independent. **(a)**. Prove that $\mathbf{x}_1(t), \ldots, \mathbf{x}_n(t)$ are linearly independent for each $t \in I$. **(b)**. Prove that $\mathbf{x} : I \to \mathbf{R}^n$ is a solution of (*) if and only if $\mathbf{x}$ is a linear combination of the $\mathbf{x}_i$:

$$\mathbf{x}(t) = \sum_{i=j}^{n} c_j \mathbf{x}_j(t)$$

for some constants $c_1, \ldots, c_j$ in $\mathbf{R}$. **(c).** Suppose that $f : I \to \mathbf{R}^n$ is continuous and that

$$p'(t) = A(t)p(t) + f(t)$$

for all $t \in I$. Show that $\mathbf{x} : I \to \mathbf{R}^n$ is a solution of

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + f(t)$$

if and only if

$$\mathbf{x}(\cdot) = p(\cdot) + \sum_{j=1}^{n} c_j \mathbf{x}_j(\cdot)$$

for some constants $c_1, \ldots, c_n \in \mathbf{R}$, where $\mathbf{x}_j(\cdot)$ are as above.

Remark: Thus

$$\left\{ \sum_{j=1}^{n} c_j \mathbf{x}_j(\cdot) : c_1, \ldots, c_n \in \mathbf{R} \right\}$$

is the set of all solutions to $x' = Ax$, and

$$\left\{ p(\cdot) + \sum_{j=1}^{n} c_j \mathbf{x}_j(\cdot) : c_1, \ldots, c_n \in \mathbf{R} \right\}$$

is the set of all solutions of $x' = Ax + f$.

2. Consider a differential equation

$$\frac{dy}{dx} = f(x, y), \tag{†}$$

where $f : \mathbf{R}^2 \to \mathbf{R}^2$. Suppose $f(x, y)$ only depends on the ratio of y to x: $f(x, y) = \phi(y/x)$ for all y and $x \neq 0$. (a). Let $v = y/x$ and, using (†), express dv/dx in terms of v and x. (b). "Solve" the resulting equation by expressing x as an integral expression involving v.

3. Solve the equation $dy/dx = (x+y)/(x-y)$. (Your answer will be an equation of the form $g(x,y) = C$ that expresses the relation between x and y.)

4. Problem 4.2 in the text.

5. Find e^{At}, where $A = \begin{bmatrix} 3 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

6. Suppose N is a nilpotent $n \times n$ matrix. Prove that $I + N$ is invertible.

7. Let A be an $n \times n$ matrix and let $p(\lambda)$ be its characteristic polynomial: $p(\lambda) = \det(\lambda I - A) = \prod_{i=1}^{k}(\lambda - \lambda_i)^{\nu_i}$, where the λ_i are distinct. Using the fact that

$$(*) \qquad \mathbf{C}^n = \oplus_{i=1}^{k} \ker((\lambda_i I - A)^{\nu_i}),$$

prove that $p(A) = 0$.