MATH 63CM HOMEWORK 4 SOLUTIONS

(a). We prove the contrapositive. Suppose that $\mathbf{x}_1, \dots, \mathbf{x}_n$ are linearly dependent at some time t_1. Then there are constants $c_1, \dots, c_n$, not all zero, such that

$$\sum c_i \mathbf{x}_i(t_1) = 0.$$

Then $x(t) := \sum c_i \mathbf{x}_i(t)$ is a solution of the initial value problem

$$\begin{aligned} \mathbf{x}'(t) &= A(t)\mathbf{x}(t), \\ \mathbf{x}(t_1) &= 0. \end{aligned}$$

The 0 function is also a solution of this initial value problem. But we know solutions are unique. Thus $\mathbf{x}(\cdot) \equiv 0$, so $\mathbf{x}_1, \dots, \mathbf{x}_n$ are linearly dependent at all times.

(a). (Alternate solution). Let $X(t)$ be the $n \times n$ matrix whose columns are $\mathbf{x}_1(t) \dots \mathbf{x}_n(t)$. Note that

$$X'(t) = A(t)X(t).$$

(That is because, by definition of matrix multiplication, column j of $A(t)X(t)$ is $A(t)\mathbf{x}_j(t)$.) Since the columns of $X(t_0)$ are independent, $\det X(t_0) \neq 0$. By Liouville's Theorem (Proposition 3.13 in the text),

$$\det X(t) = \det X(t_0)\, e^{\int_{t_0}^t \operatorname{tr} A(s)\, ds}.$$

Thus $\det X(t) \neq 0$ for all t, so the columns of $X(t)$ are independent for each t.

(b). Trivially, each linear combination of the $\mathbf{x}_i$ is a solution:

$$\frac{d}{dt} \sum_i c_i \mathbf{x}_i(t) = \sum c_i \mathbf{x}_i'(t) = \sum c_i A \mathbf{x}_i(t) = A\left(\sum_i c_i \mathbf{x}_i(t)\right).$$

To see that we get all solutions in this way, let $\mathbf{x}(\cdot)$ be any solution of the equation. Since $\mathbf{x}_1(t_0), \dots, \mathbf{x}_n(t_0)$ are n independent vectors in $\mathbf{R}^n$, they form a basis for $\mathbf{R}^n$. Thus $\mathbf{x}(t_0)$ is a linear combination of the $\mathbf{x}_i(t_0)$:

$$\mathbf{x}(t_0) = \sum_i c_i \mathbf{x}_i(t_0)$$

for suitable constants $c_1, \dots, c_n$. Now $\mathbf{x}(t)$ and $\sum_i c_i \mathbf{x}_i(t)$ are two solutions of the ODE that are equal at time t_0. By the uniqueness theorem, they are equal for all t.

2. We have

$$\frac{dv}{dx} = \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = \frac{\phi(v) - v}{x}.$$

By separation of variables, we therefore have

$$\int \frac{dv}{\phi(v) - v} = \int \frac{dx}{x} = \log|x| + C,$$

so

$$|x| = \tilde{C} \exp\left\{\int \frac{dv}{\phi(v) - v}\right\}.$$

3. Ths falls under the framework of the previous problem with

$$\phi(v) = \frac{1+v}{1-v}.$$

Thus we have

$$\begin{aligned}
|x| &= \tilde{C} \exp\left\{\int \frac{dv}{\frac{1+v}{1-v} - v}\right\} \\
&= \tilde{C} \exp\left\{\int \frac{1-v}{1+v-v(1-v)} dv\right\} \\
&= \tilde{C} \exp\left\{\int \frac{1-v}{1+v^2} dv\right\} \\
&= \tilde{C} \exp\left\{\arctan v - \frac{1}{2} \log|1+v^2|\right\} \\
&= \tilde{C} \frac{\exp\{\arctan(y/x)\}}{\sqrt{1+(y/x)^2}} \\
&= \tilde{C}|x| \frac{\exp\{\arctan(y/x)\}}{\sqrt{x^2+y^2}}.
\end{aligned}$$

Therefore, we have for $x \neq 0$

$$\tilde{C} \frac{\exp\{\arctan(y/x)\}}{\sqrt{x^2+y^2}} = 1.$$

4. (a). The equilibrium points are when $x_2^2 + x_1 x_2 = 2$ and $x_1^2 + x_1 x_2 = 2$. Adding and subtracting the two equations, we see that this occurs when $(x_1 + x_2)^2 = 4$ and $x_1^2 = x_2^2$. The solutions to this system are $(x_1, x_2) = (1, 1)$ and $(x_1, x_2) = (-1, -1)$.

Around $(1, 1)$, the linearized equation is

$$\begin{aligned}
\tilde{x}_1' &= \tilde{x}_1 + 3\tilde{x}_2 \\
\tilde{x}_2' &= 3\tilde{x}_1 + \tilde{x}_2;
\end{aligned}$$

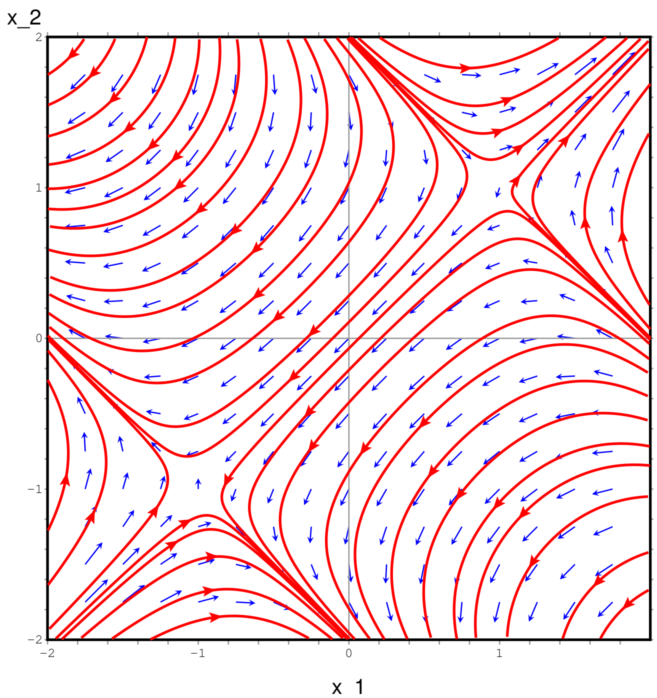
the eigenvalues of this system are -2 and 4, so this is a hyperbolic equilibrium point and we have a saddle.

Around $(-1, -1)$, the linearized equation is

$$\begin{aligned}
\tilde{x}_1' &= -\tilde{x}_1 - 3\tilde{x}_2 \\
\tilde{x}_2' &= -3\tilde{x}_1 - \tilde{x}_2;
\end{aligned}$$

the eigenvalues of this system are 2 and -4, so this is a hyperbolic equilibrium point and we have a saddle.

(b). The phase portrait is here:



(c). We have

$$(x_1 + x_2)' = (x_1 + x_2)^2 - 4$$
$$(x_1 - x_2)' = (x_1 + x_2)(x_1 - x_2)$$

From this we see that the curve $x_1 = x_2$ is invariant under the evolution. Thus it is the unstable manifold for the equilibrium point $(1, 1)$ and the stable manifold for the equilibrium point $(-1, -1)$. (The stability can be checked by checking the sign of the derivative along the manifold.) We also see that the curves $x_1 + x_2 = 2$ and $x_1 + x_2 = -2$ are invariant under the evolution. Therefore, the curve $x_1 + x_2 = 2$ is the stable manifold for the equilibrium point $(1, 1)$, and $x_1 + x_2 = -2$ is the unstable manifold for the equilibrium point $(-1, -1)$.

5. Note that

$$\begin{aligned} \det(\lambda I - A) &= \begin{vmatrix} \lambda - 3 & 1 \\ -1 & \lambda - 1 \end{vmatrix} (\lambda - 2) \\ &= ((\lambda - 3)(\lambda - 1) + 1)(\lambda - 2) \\ &= (\lambda^2 - 4\lambda + 4)(\lambda - 2) \\ &= (\lambda - 2)^3. \end{aligned}$$

Consequently the matrix $N = A - 2I = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ is nilpotent (indeed $N^3 = 0$) and commutes with A. Thus

$$\begin{aligned}
e^{At} &= e^{t(2I+N)} \\
&= e^{2tI}e^{tN} \\
&= e^{2t}\left(I + tN + \frac{t^2}{2!}N^2\right) \\
&= e^{2t}\left(I + t\begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \frac{t^2}{2}\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}\right) \\
&= e^{2t}\begin{bmatrix} (1+t) & -t & -t^2/2 \\ t & (1-t) & t \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

6. Let k be a positive integer such that $N^k = 0$. Then

$$(I+N)(I - N + N^2 - \dots N^{k-1}) = I - N^k = I.$$

7. Let $\mathbf{x} \in \mathbf{C}^n$. Then (by (*) in the statement of the problem), there exist $\mathbf{x}_i \in \ker(\lambda_i I - A)^{\nu_i}$ such that

$$\mathbf{x} = \mathbf{x}_1 + \cdots + \mathbf{x}_k.$$

Note that if $q(z)$ and $\hat{q}(z)$ are two polynomials, then $q(A)\hat{q}(A) = \hat{q}(A)q(A)$. Thus

$$\begin{aligned}
p(A)\mathbf{x}_j &= \prod_{i=1}^{k}(\lambda_i I - A)^{\nu_i}\mathbf{x}_j \\
&= \left(\prod_{i\neq j}(\lambda_i I - A)^{\nu_i}\right)(\lambda_j I - A)^{\nu_j}\mathbf{x}_j \\
&= 0.
\end{aligned}$$

Thus

$$p(A)\mathbf{x} = p(A)\left(\sum_{j=1}^{k}\mathbf{x}_j\right) = \sum_{j=1}^{k}p(A)\mathbf{x}_j = 0.$$

We have shown that $p(A)\mathbf{x} = 0$ for every vector $\mathbf{x}$. Thus $p(A) = 0$.