Math 63CM Spring 2019 Homework 5 Solutions

1 **Solution**: Let $M = \begin{bmatrix} A & B \\ \mathbf{0} & C \end{bmatrix}$. Case 1: A and C are upper triangular. Then M is also upper triangular. Recall that the determinant of a triangular matrix is the product of its diagonal elements. In this case, the result follows immediately.

Case 2: general A and C. We know there are square matrices S and T such that $S^{-1}AS$ and $T^{-1}CT$ are upper triangular. Let

$$Q = \begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix}.$$

Then

$$Q^{-1}MQ = \begin{bmatrix} S^{-1}AS & S^{-1}BT \\ 0 & T^{-1}CT \end{bmatrix}.$$

Thus

$$\det(Q^{-1}MQ) = \det(S^{-1}AS)\det(T^{-1}CT)$$

by case 1. But $\det(Q^{-1}MQ) = \det(M)$, $\det(S^{-1}AS) = \det A$, and $\det(T^{-1}CT) = \det C$, so we are done.

Alternate Solution: If A is not invertible, then there is a $v \in \mathbf{R}^k$ so that $Av = 0$, so

$$\begin{pmatrix} A & B \\ \mathbf{0} & C \end{pmatrix}\begin{pmatrix} v \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} Av \\ \mathbf{0} \end{pmatrix} = \mathbf{0},$$

so

$$\det\begin{pmatrix} A & B \\ \mathbf{0} & C \end{pmatrix} = 0 = (\det A)(\det C).$$

Therefore, we may assume that A and C are invertible. Then we have

$$\begin{pmatrix} A & B \\ & C \end{pmatrix}\begin{pmatrix} I & -A^{-1}B \\ & I \end{pmatrix} = \begin{pmatrix} A & \\ & C \end{pmatrix}.$$

Since $\begin{pmatrix} I & -A^{-1}B \\ & I \end{pmatrix}$ is upper-triangular with 1s on the diagonal, we have that

$$\det\begin{pmatrix} I & -A^{-1}B \\ & I \end{pmatrix} = 1.$$

Also, we have that

$$\det\begin{pmatrix} A & \\ & C \end{pmatrix} = (\det A)(\det C),$$

which completes the proof.

2 If (x, y) is a critical point, then either $x = 1$ or $x = -y$. If $x = 1$ then we must have $y = 1$, and if $x = -y$ then we must have $y = y^2$, so $y \in \{0, 1\}$. Thus there are three critical points, namely $(0, 0)$, $(1, 1)$, and $(-1, 1)$.

At $(0, 0)$, the linearized equation is

$$\tilde{x}' = -\tilde{x} - \tilde{y}$$
$$\tilde{y}' = \tilde{y}.$$

This linear system corresponds to the matrix

$$\begin{pmatrix} -1 & -1 \\ & 1 \end{pmatrix},$$

which has eigenvalues ± 1, so the system is hyperbolic, and in fact a saddle. Thus in particular it's an unstable equilibrium.

At $(1,1)$, in order to linearize the system we first change coordinates by $x = 1 + X$ and $y = 1 + Y$. Then we have

$$X' = X(2 + X + Y)$$
$$Y' = 1 + Y - (1 + X)^2.$$

The linearization is given by

$$\tilde{X}' = 2\tilde{X}$$
$$\tilde{Y}' = \tilde{Y} - 2\tilde{X}.$$

This corresponds to the matrix

$$\begin{pmatrix} 2 & \\ -2 & 1 \end{pmatrix},$$

which has eigenvalues $2, 1$. Hence it's a hyperbolic equilibrium point which is a source, and hence the equilibrium point is unstable.

At $(-1,1)$, we change coordinates by $x = -1 + X$ and $y = 1 + Y$. Then we have

$$X' = (X - 2)(X + Y)$$
$$Y' = 1 + Y - (-1 + X)^2.$$

The linearization is given by

$$\tilde{X}' = -2\tilde{X} - 2\tilde{Y}$$
$$\tilde{Y}' = \tilde{Y} + 2\tilde{X}.$$

This corresponds to the matrix

$$\begin{pmatrix} -2 & -2 \\ 2 & 1 \end{pmatrix}.$$

To find the eigenvalues of this matrix, we set the char poly equal to 0:

$$0 = (-2 - \lambda)(1 - \lambda) + 4 = \lambda^2 + \lambda + 2,$$

so the eigenvalues are $\lambda = \frac{-1 \pm \mathrm{i}\sqrt{-7}}{2}$. The real parts of both eigenvalues are negative, so again we have a hyperbolic equilibrium point which is a spiral sink, hence stable and asymptotically stable.

3 We look for a Lyapunov function of the form $L(x,y) = ax^2 + by^4$. From the ODE we get that

$$\frac{\mathrm{d}}{\mathrm{d}t} L(x(t), y(t)) = 2ax\dot{x} + 4by^3\dot{y} = -4axy^3 + 4by^3(x - 3y^3) = -4axy^3 + 4by^3x - 12by^4.$$

Take $a = b = 1$, so we have

$$\frac{\mathrm{d}}{\mathrm{d}t} L(x(t), y(t)) = -12by^4 \leq 0.$$

On the other hand, L has a local minimum at 0, so by Lyapunov's first theorem we see that $(0,0)$ is a stable equilibrium.

We also note that the set K of (x,y) for which $\frac{\mathrm{d}}{\mathrm{d}t} L(x(t), y(t)) = 0$ is given by the set $\{y = 0\}$. We note that on this set, we have $y' \neq 0$ unless $x = 0$, so no point in K except for the origin remains in K under the flow. Therefore, by the LaSalle/Krasovski theorem, $(0,0)$ is asymptotically stable.

4 If $\delta < 0$, then the eigenvalues of A are both real and have opposite signs, so the origin is an unstable equilibrium.

If $\delta > 0$ and $\tau > 0$, then the eigenvalues of A both have positive real part, so the origin is an unstable equilibrium.

If $\delta > 0$ and $\tau < 0$, then the eigenvalues of A both have negative real part, so the origin is an asymptotically-stable equilibrium.

If $\delta > 0$ and $\tau = 0$, then the eigenvalues of A are both nonzero and imaginary, so of the form $\pm\mathrm{i}c$ for $c \in \mathbf{R}$. Then we have a matrix C so that

$$\exp(tA) = C \begin{pmatrix} \mathrm{e}^{\mathrm{i}ct} & \\ & \mathrm{e}^{-\mathrm{i}ct} \end{pmatrix} C^{-1},$$

so the origin is a stable but not asymptotically stable equilibrium.

If $\delta = 0$ and $\tau > 0$, then one of the eigenvalues of A is zero and the other is positive, so the origin is an unstable equilibrium.

If $\delta = 0$ and $\tau < 0$, then one of the eigenvalues of A is zero and the other is negative. Let λ be the negative eigenvalue. Then we have a matrix C so that

$$\exp(tA) = C\begin{pmatrix} 1 & \\ & e^{\lambda t} \end{pmatrix} C^{-1},$$

so the origin is a stable but not asymptotically stable equilibrium.

If $\delta = \tau = 0$, then both eigenvalues are 0. In this case the origin may or may not be a stable equilibrium. For example, if $A = 0$ then the origin is a stable but not asymptotically-stable equilibrium, but if $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then the origin is an unstable equilibrium, since

$$\exp(tA) = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}.$$

However, in this case the origin is never an asymptotically-stable equilibrium, since there is a vector $x \neq 0$ so that $Ax = 0$.

5 We note that the char poly of A is

$$((1-\lambda)(-1-\lambda)+2)^2$$

by problem 1. We set this equal to 0 to obtain

$$0 = ((1-\lambda)(-1-\lambda)+2)^2 = (\lambda^2+1)^2.$$

Therefore, the eigenvalues are $\lambda = \pm \mathrm{i}$, each with algebraic multiplicity 2.

We have

$$(A-\mathrm{i})^2 = \begin{pmatrix} -2-2\mathrm{i} & 2\mathrm{i} & -2 & 2-2\mathrm{i} \\ -4\mathrm{i} & -2+2\mathrm{i} & -2-2\mathrm{i} & 4 \\ 0 & 0 & -2+2\mathrm{i} & -4\mathrm{i} \\ 0 & 0 & 2\mathrm{i} & -2-2\mathrm{i} \end{pmatrix}.$$

The kernel of this matrix is spanned by $(1+\mathrm{i}, 2, 0, 0)$ and $(0, 0, \mathrm{i}-1, -1)$. Also, we have

$$(A+\mathrm{i})^2 = \begin{pmatrix} -2+2\mathrm{i} & -2\mathrm{i} & -2 & 2+2\mathrm{i} \\ 4\mathrm{i} & -2-2\mathrm{i} & -2+2\mathrm{i} & 4 \\ 0 & 0 & -2-2\mathrm{i} & 4\mathrm{i} \\ 0 & 0 & -2\mathrm{i} & -2+2\mathrm{i} \end{pmatrix}.$$

The kernel of this matrix is spanned by $(-1+\mathrm{i}, -2, 0, 0)$ and $(0, 0, 1+\mathrm{i}, 1)$. By the algorithm given in the book, we can take

$$\begin{aligned} L &= \begin{pmatrix} 1+\mathrm{i} & 0 & -1+\mathrm{i} & 0 \\ 2 & 0 & -2 & 0 \\ 0 & \mathrm{i}-1 & 0 & 1+\mathrm{i} \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{i} & & & \\ & \mathrm{i} & & \\ & & -\mathrm{i} & \\ & & & -\mathrm{i} \end{pmatrix} \begin{pmatrix} 1+\mathrm{i} & 0 & -1+\mathrm{i} & 0 \\ 2 & 0 & -2 & 0 \\ 0 & \mathrm{i}-1 & 0 & 1+\mathrm{i} \\ 0 & -1 & 0 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & -1 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -1 & 1 \end{pmatrix}. \end{aligned}$$

Then we have

$$N = A - L = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & & & \\ & & & \end{pmatrix},$$

which is nilpotent.

(We could have also noticed that A is block-upper-triangular with diagonalizable blocks, and then we wouldn't have had to compute the eigenvalues.)

6 We use the method of variation of parameters. First we find the matrix solution to the homogeneous problem

$$\begin{aligned} A'(t) &= \begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix} A(t) \\ A(0) &= I. \end{aligned}$$

The solution is of course

$$A(t) = e^{tA},$$

so we want to compute e^{tA}. The characteristic polynomial of A is

$$(2-\lambda)(-1-\lambda) - 4 = -2 + \lambda - 2\lambda + \lambda^2 - 4 = \lambda^2 - \lambda - 6,$$

which has roots $\lambda = 3$ and $\lambda = -2$. The eigenvector corresponding to $\lambda = 3$ is $(1, 1)$ the eigenvector corresponding to $\lambda = -2$ is $(1, -4)$. Therefore, we have

$$A(t) = e^{tA} = \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} e^{3t} & \\ & e^{-2t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}^{-1}.$$

Now we look for a solution of our desired problem

$$x'(t) = \begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 5t \end{pmatrix}$$

of the form

$$x(t) = A(t)c(t).$$

We have

$$\begin{aligned} x'(t) &= A'(t)c(t) + A(t)c'(t) \\ &= \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} A(t)c(t) + A(t)c'(t) \\ &= \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} x(t) + A(t)c'(t). \end{aligned}$$

So to solve our problem we would want

$$\begin{aligned} c'(t) &= A(t)^{-1} \begin{pmatrix} 0 \\ 5t \end{pmatrix} \\ c(0) &= x(0). \end{aligned}$$

We can solve this equation by

$$\begin{aligned} c(t) &= x(0) + \int_0^t A(s)^{-1} \begin{pmatrix} 0 \\ 5s \end{pmatrix} ds \\ &= x(0) + \int_0^t \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} e^{-3s} & \\ & e^{2s} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 5s \end{pmatrix} ds \\ &= x(0) + \int_0^t \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} e^{-3s} & \\ & e^{2s} \end{pmatrix} \frac{1}{5} \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 5s \end{pmatrix} ds \\ &= x(0) + \int_0^t \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} e^{-3s} & \\ & e^{2s} \end{pmatrix} \begin{pmatrix} s \\ -s \end{pmatrix} ds \\ &= x(0) + \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} \frac{1}{9}\left(1 - e^{-3t}(1+3t)\right) \\ \frac{1}{4}\left(-1 + e^{2t}(1-2t)\right) \end{pmatrix}. \end{aligned}$$

Therefore, we have

$$
\begin{aligned}
x(t) &= \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{3t} & \\ & \mathrm{e}^{-2t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}^{-1} x(0) + \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{3t} & \\ & \mathrm{e}^{-2t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} \frac{1}{9}\left(1 - \mathrm{e}^{-3t}(1+3t)\right) \\ \frac{1}{4}\left(-1 + \mathrm{e}^{2t}(1-2t)\right) \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{3t} & \\ & \mathrm{e}^{-2t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}^{-1} x(0) + \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} \frac{1}{9}\left(\mathrm{e}^{3t} - (1+3t)\right) \\ \frac{1}{4}\left(-\mathrm{e}^{-2t} + 1 - 2t\right) \end{pmatrix} \\
&= \frac{1}{5} \begin{pmatrix} \mathrm{e}^{-2t} + 4\mathrm{e}^{3t} & -\mathrm{e}^{-2t} + \mathrm{e}^{3t} \\ -4\mathrm{e}^{-2t} + 4\mathrm{e}^{3t} & 4\mathrm{e}^{-2t} + \mathrm{e}^{3t} \end{pmatrix} x(0) + \begin{pmatrix} \frac{1}{9}\left(-1 + \mathrm{e}^{3t} - 3t\right) + \frac{1}{4}\left(1 - \mathrm{e}^{-2t} - 2t\right) \\ -1 + 2t + \mathrm{e}^{-2t} + \frac{1}{9}\left(-1 + \mathrm{e}^{3t} - 3t\right) \end{pmatrix}.
\end{aligned}
$$