MATH 63CM HOMEWORK 6

DUE 11:59PM ON SUNDAY, MAY 19

1. While on a Star Trek mission, you are beamed into a two-dimensional universe H consisting of the points $\{(x, y) : y > 0\}$. Mysteriously, as you approach the x-axis, you and all of your measuring instruments shrink: if you measure the infinitesimal segment from (x, y) to $(x + dx, y + dy)$, instead of getting $\sqrt{dx^2 + dy^2}$, you get $\frac{\sqrt{dx^2+dy^2}}{y}$. Thus (in this universe) the length of a curve $s \in [a, b] \mapsto (x(s), y(s))$ is given by

$$\int_a^b \frac{\sqrt{x'(s)^2 + y'(s)^2}}{y}\, ds.$$

Using calculus of variations, find the shortest curve joining a pair of points in H. (You may look for curves given by $y = y(x)$ or curves given by $x = x(y)$. You do not have to prove that the solution you find is in fact a minimum.)

Describe the shape of your solution curves geometrically.

2. Consider the system

$$x' = -2x - y^2,$$
$$y' = -y - x^2.$$

(a). Prove that $(0, 0)$ is an asymptotically stable equilibrium.
(b). Find an $r > 0$ as large as you can such that any solution that starts in the ball of radius r about the origin converges to $(0, 0)$ as $t \to \infty$.

3. Consider n particles on a line. Let m_i be the mass of the ith particle and $x_i(t)$ be its position at time t. Suppose there is a potential energy function $V : \mathbf{R}^n \to \mathbf{R}$ such that the force acting on particle i is $-\frac{\partial V}{\partial x_i}(x_1(t), \ldots, x_n(t))$. The kinetic energy is $K(\dot{x}_1, \ldots, \dot{x}_n) = \sum_{i=1}^n \frac{1}{2} m_i \dot{x}_i^2$. Consider the problem of minimizing

$$\int_{t_0}^{t_1} (K(\dot{\mathbf{x}}(t)) - V(\mathbf{x}(t))\, dt$$

among paths $\mathbf{x} : [t_0, t_1] \to \mathbf{R}^n$ joining two specified points: $\mathbf{x}(t_0) = \mathbf{x}_0$ and $\mathbf{x}(t_1) = \mathbf{x}_1$. Write the differential equations that such an $\mathbf{x}(\cdot)$ would have to satisfy.

Remark: The problem asks about particles moving on a line, but it applies to N particles moving in $\mathbf{R}^3$ as follows. Let $(x_1(t), x_2(t), x_3(t))$ be the position of the first particle and let $m_1 = m_2 = m_3$ be its mass. Let $(x_4(t), x_5(t), x_6(t))$ be the position of the second particle and let $m_4 = m_5 = m_6$ be its mass, and so on. Thus the system of N particles moving in $\mathbf{R}^3$ may be thought of as a system of $3N$ particles moving on a line.