Math 63CM Spring 2019 Homework 6 Solutions

1. While on a Star Trek mission, you are beamed into a two-dimensional universe H consisting of the points $\{(x,y) : y > 0\}$. Mysteriously, as you approach the x-axis, you and all of your measuring instruments shrink: if you measure the infinitesimal segment from (x,y) to $(x+dx, y+dy)$, instead of getting $\sqrt{dx^2+dy^2}$, you get $\frac{\sqrt{dx^2+dy^2}}{y}$. Thus (in this universe) the length of a curve $s \in [a,b] \mapsto (x(s), y(s))$ is given by

$$\int_a^b \frac{\sqrt{x'(s)^2 + y'(s)^2}}{y}\, ds.$$

Using calculus of variations, find the shortest curve joining a pair of points in H. (You may look for curves given by $y = y(x)$ or curves given by $x = x(y)$. You do not have to prove that the solution you find is in fact a minimum.)

Describe the shape of your solution curves geometrically.

Solution: Let's look for solutions of the form $y = y(x)$. We wish to minimize

$$\begin{aligned} \mathcal{L}[y(\cdot)] &= \int \frac{\sqrt{dx^2+dy^2}}{y} \\ &= \int \frac{\sqrt{1+(\dot{y})^2}}{y}\, dx \end{aligned}$$

where $\dot{y} = \frac{dy}{dx}$. Thus

$$L(x,y,\dot{y}) = \frac{\sqrt{1+(\dot{y})^2}}{y},$$

Since L does not depend on x, the Euler-Lagrange Equation simplifies to

$$\begin{aligned} c &= \dot{y}\frac{\partial L}{\partial \dot{y}} - L \\ &= \dot{y}\frac{\dot{y}}{y\sqrt{1+\dot{y}^2}} - \frac{\sqrt{1+\dot{y}^2}}{y} \\ &= \frac{-1}{y\sqrt{1+\dot{y}^2}}. \end{aligned}$$

Squaring and solving for $\dot{y}^2$ gives

$$\dot{y}^2 = \frac{1-c^2y^2}{c^2y^2}.$$

Thus

$$\frac{dx}{dy} = \pm\frac{cy}{\sqrt{1-c^2y^2}},$$

so

$$x = \pm\int \frac{cy}{\sqrt{1-c^2y^2}}\, dy + k$$

Letting $y = \frac{1}{c}\sin\theta$, we have

$$\begin{aligned} x &= \pm\int \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\,\frac{1}{c}\cos\theta\, d\theta + k \\ &= \pm\frac{1}{c}\int \sin\theta\, d\theta + k \\ &= \pm\frac{1}{c}\cos\theta + k. \end{aligned}$$

Thus the solution curves are given by

$$\begin{aligned} x &= \frac{1}{c}\cos\theta + k, \\ y &= \frac{1}{c}\sin\theta. \end{aligned}$$

In the problem, we are given that $y > 0$. Thus we may assume that $c > 0$ and that $\theta \in (0,\pi)$. Letting $r = \frac{1}{c}$, we rewrite the equations as

$$\begin{aligned} x &= r\cos\theta + k, \\ y &= r\sin\theta. \end{aligned}$$

Note that this is the equation of the semicircle of radius r centered at a point on the x-axis (namely, the point $(k,0)$.)

Remark. For any two points (x_0, y_0) and (x_1, y_1) in H with $x_0 \neq x_1$, there is exactly one such semicircle containing both points. It is possible to show that the arc of the semicircle between them is indeed the (unique) shortest path joining them.

What if $x_0 = x_1$? Then there is no path of the form $y = y(x)$ joining them. In this case, we can look for paths of the form $x = x(y)$. If you solve the Euler-Lagrange Equation, you will see that the solutions include $x =$ constant. In particular, if $x_0 = x_1$, then the straight line segment joining (x_0, y_0) and (x_1, y_1) is a solution to the Euler-Lagrane Equation. It is possible to show that that segment is the unique shortest path joining those two points.

Thus in this (non-Euclidean) geometry, the geodesics (i.e., the analogs of straight lines) are: semicirces centered at points on the x-axis, and vertical rays of the form $x =$ constant. (Note that the vertical ray $x = c$ is the limit of the semicircle $(x-k)^2 + y^2 = (c-k)^2$ as $k \to \pm\infty$. Thus such a vertical ray may be thought of as a semicircle whose center is infinitely far away.)

2

a We note that the linearization of the ODE around $(0,0)$ is

$$\begin{aligned} \tilde{x}' &= -2\tilde{x} \\ \tilde{y}' &= -\tilde{y}. \end{aligned}$$

The eigenvalues of this system are -2 and -1, so the system has an asymptotically stable equilibrium since both real parts are strictly less than 0.

b Define $L(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$. We notice that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}L(x(t),y(t)) = xx' + yy' &= -2x^2 - xy^2 - y^2 - x^2y \\ &\leq -(x^2+y^2) - xy(x+y). \end{aligned}$$

Now we have $|xy| \leq \frac{x^2+y^2}{2}$ and $|x| + |y| \leq \sqrt{(|x|+|y|)^2} = \sqrt{x^2 + 2|xy| + y^2} \leq \sqrt{2(x^2+y^2)}$, so

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}L(x(t),y(t)) &\leq -(x^2+y^2) + \frac{x^2+y^2}{2}\sqrt{2(x^2+y^2)} \\ &= -(x^2+y^2)\left[1 - \frac{\sqrt{2}}{2}\sqrt{x^2+y^2}\right]. \end{aligned}$$

Therefore, if $\sqrt{x^2+y^2} < \sqrt{2}$, then L is strictly decreasing, so by the Lyapunov theorems the basin of attraction of $(0,0)$ is at least the set $\{r < \sqrt{2}\}$.

3 We must have, for all $\boldsymbol{\eta}$ with $\boldsymbol{\eta}(t_0) = \boldsymbol{\eta}(t_1) = 0$,

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}\delta} \int_{t_0}^{t_1} (K(\dot{\mathbf{x}}(t) + \delta\dot{\boldsymbol{\eta}}(t)) - V(\mathbf{x}(t) + \delta\boldsymbol{\eta}(t)))\,\mathrm{d}t \\
&= \int_{t_0}^{t_1} (\nabla K(\dot{\mathbf{x}}(t)) \cdot \dot{\boldsymbol{\eta}}(t) - \nabla V(\mathbf{x}(t)) \cdot \boldsymbol{\eta}(t))\,\mathrm{d}t \\
&= \int_{t_0}^{t_1} \left(\sum_{i=1}^{n} m_i \dot{x}_i(t) \dot{\eta}_i(t) - \nabla V(\mathbf{x}(t)) \cdot \boldsymbol{\eta}(t) \right) \mathrm{d}t \\
&= \int_{t_0}^{t_1} \left(-\sum_{i=1}^{n} m_i \ddot{x}_i(t) \eta_i(t) - \nabla V(\mathbf{x}(t)) \cdot \boldsymbol{\eta}(t) \right) \mathrm{d}t \\
&= \int_{t_0}^{t_1} \left(-\sum_{i=1}^{n} m_i \ddot{x}_i(t) \eta_i(t) - \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(\mathbf{x}(t)) \eta_i(t) \right) \mathrm{d}t \\
&= \int_{t_0}^{t_1} \sum_{i=1}^{n} \eta_i(t) \left(-m_i \ddot{x}_i(t) - \frac{\partial V}{\partial x_i}(\mathbf{x}(t)) \right) \mathrm{d}t.
\end{aligned}$$

Since this must hold for all $\boldsymbol{\eta}$ with $\boldsymbol{\eta}(t_0) = \boldsymbol{\eta}(t_1) = 0$, by the fundamental lemma of the calculus of variations we must have

$$m_i \ddot{x}_i(t) = -\frac{\partial V}{\partial x_i}(\mathbf{x}(t)),$$

which is Newton's second law.