HW 7 SOLUTIONS

1. Write $z = x + \imath y$, where $\imath^2 = -1$. Then we have

$$\begin{aligned} z' &= x' + \imath y' \\ &= 3x - y - xe^{x^2+y^2} + \imath\left(x + 3y - ye^{x^2+y^2}\right) \\ &= (3+\imath)x + (3+\imath)\imath y - (x+\imath y)e^{x^2+y^2} \\ &= (3+\imath)z - ze^{|z|^2}. \end{aligned}$$

Now we have $(x, y) \cdot (x', y') = \Re(z'\overline{z}) = \Re\left((3+\imath)|z|^2 - |z|^2 e^{|z|^2}\right) = |z|^2\left(3 - e^{|z|^2}\right)$.
Solution 1: Therefore, if $|z|$ is so small that $3 - e^{|z|^2} > 0$, then $(x, y) \cdot (x', y') > 0$, while if $|z|$ is large enough that $3 - e^{|z|^2} < 0$, then $(x, y) \cdot (x', y') < 0$. This means that there exist $0 < r < R < \infty$ so that the annulus $A = \{r < |(x, y)| < R\}$ is positively invariant under the flow. Since A is compact, the flow started in A must have an ω-limit set. But there are no equilibrium points inside A, so this limit set must be a closed orbit by the Poincar–Bendixson theorem.
Solution 2: Note that if $|z| = \sqrt{\log 3}$, then $z' = \imath z$. Thus we can find explicit solution $z = \sqrt{\log 3}e^{\imath t}$, which is a closed orbit.

From the text:

.1. Let f be a fixed polynomial such that $f' = p$. Consider the quantity $V(x, y) = f(x) + \frac{1}{2}y^2$, then the following holds.

$$\frac{d}{dt}\left(f(x) + \frac{1}{2}y^2\right) = f'(x)x' + yy' = p(x)y + y(-y^3 - p(x)) = -y^4 \leq 0,$$

hence $V(x, y)$ is monotonically decreasing over time. Since f has even degree and positive leading coefficient, we get $V(x, y) \to \infty$ whenever $||(x, y)|| \to \infty$, hence this implies that for any solution to the ODE at hand, $||(x(t), y(t))||$ and hence also $||(x(t), y(t))||^2 = x(t)^2 + y(t)^2$ stays bounded for all t, as claimed.

(ii) If not, there would be a finite time T_+ such that for $t \nearrow T_+$, we would get that $||(x(t), y(t))|| \to \infty$. Since this does not happen (see (i)), we deduce that the ODE has a solution for all time.

(iii) Since the quantity $V(x, y)$ from (i) is monotonically decreasing over time, it has to attain a local minimum at every ω-limit point $(\bar{x}, \bar{y})$, hence both partial derivatives $\frac{\partial V}{\partial x} = p(x)$ and $\frac{\partial V}{\partial y} = y$ have to vanish at $(\bar{x}, \bar{y})$, hence $p(\bar{x}) = \bar{y} = 0$.

(iv) By (i) and (ii), we know that $(x(t), y(t))$ exists for all time and stays bounded, hence the sequence $(x(n), y(n))$ with $n = 1, 2, 3, \ldots$ has all its entries in a compact disk, thus it has a converging subsequence with limit $(\bar{x}, \bar{y})$. It follows from the definition of ω-limit points that $(\bar{x}, \bar{y})$ is one, hence we have proved that the ω-limit set is non-empty.
Now by (iii), we know that there are at most $\deg(p)$ points which could be an ω-limit points, namely the points $(x_i, 0)$ with $x_1, \ldots, x_r$ the roots of p. Now for any of these points which are local minima

of V with positive Hessian matrix, we can choose $\epsilon > 0$ and $K > 1$ such that

$$\sup_{(x,y)\in B_{\varepsilon/K}(x_i,y_i)} V(x,y) \quad < \quad \inf_{(x,y)\in B_\varepsilon(x_i,y_i)\setminus B_{\varepsilon/2}(x_i,y_i)} V(x,y)$$

and moreover, the balls $B_\varepsilon(x_i, y_i), i = 1, \dots, r$ are pairwise disjoint. Then if (x_j, y_j) is an ω-limit point for a particular trajectory, eventually there will be a time t such that $(x(t), y(t)) \in B_{\varepsilon/K}(x_j, y_j)$, but then (since V is monotonically decreasing), the trajectory will never leave $B_{\varepsilon/2}(x_j, y_j)$ and hence none of the other points $(x_i, y_i), i \neq j$ can be an ω-limit point, hence there is exactly one.

.2. (i) Since 0 is an ω-limit point, there exist arbitrary large t for which $||x(t)|| \leq r$ and $\langle x(t), v\rangle \leq \lambda$, hence $\tau_k \to \infty$. Fix an arbitrary $\varepsilon > 0$, with $\varepsilon < r - \lambda$. Then by definition of τ_k, there exists an k_0 such that $\langle x(\tau_k), v\rangle > \lambda - \varepsilon/2$ for all $k \geq k_0$. Since, by Proposition 5.6 in the textbook, we also know that there exists a k_1 such that $\mathrm{dist}(x(\tau_k), \Omega) < \varepsilon/2$ for all $k > k_1$, we deduce that for all $k \geq \max(k_0, k_1)$, we have

$$x(\tau_k) \in \{x \in \mathbb{R}^n : \lambda - \varepsilon/2 < \langle x(\tau_k), v\rangle \leq \lambda\} \cap \{x \in \mathbb{R}^n : \mathrm{dist}(x(\tau_k), \Omega) < \varepsilon/2\} \subset B_\varepsilon(\lambda v),$$

hence we indeed have $x(\tau_k) \to \lambda v$. The last statement in fact only holds for k large enough, we need that $\langle x(t), v\rangle$ is monotonically increasing in a neighbourhood of τ_k, which holds for k big enough, as can be seen from combining Proposition 5.6. with the definition of the points s_k (this is obvious from a picture). If we now had $\langle F(x(\tau_k)), v\rangle < 0$ for such a k, we would get that $\langle x(t), v\rangle$ is monotonically decreasing in a neighborhood of $t = \tau_k$, which contradicts our explanations above, hence we indeed have $\langle F(x(\tau_k)), v\rangle \geq 0$ for all k big enough. Combining everything, we deduce the final assertion:

$$\langle F(\lambda v), v\rangle = \lim_{k\to\infty} \langle F(x(\tau_k), v\rangle \geq 0$$

(ii) This is completely analogous to (i).

(iii) Suppose there exists some $\lambda \in (0, r)$ such that $w = F(\lambda v) \neq 0$. By parts (i) and (ii) we know that w and v are orthogonal. Choose an $\varepsilon > 0$ with $\varepsilon < r - \lambda$ such that the function $G(x) := F(x) - w$ satisfies $||G(x)|| < \frac{||w||}{2}$ throughout $B_\varepsilon(\lambda v)$. Since λv is an ω-limit point, there exists arbitrarily large t_0 such that $x(t_0) \in B_{\varepsilon/2}(\lambda v)$. Since the set Ω contains all of the line $[0, v]$, the path eventually leaves the ball $B_\varepsilon(\lambda v)$ again, so the following number is finite.

$$t_1 = \sup_{t\geq t_0}\{t \in [t_0, \infty) : \forall s \in [t_0, t) : x(s) \in B_\varepsilon(\lambda v)\}$$

Let us now set $T = t_1 - t_0$, we then have

$$x(t_1) - x(t_0) = \int_{t_0}^{t_1} F(x(t))dt = Tw + \int_{t_0}^{t_1} G(x(t))dt,$$

hence we also get

$$\begin{aligned}
\langle x(t_1) - x(t_0), w\rangle &= \langle Tw + \int_{t_0}^{t_1} G(x(t))dt, w\rangle \\
&\geq T||w||^2 - T \sup_{t\in[t_0,t_1]} ||G(x(t))|| \cdot ||w|| \\
&\geq T||w||^2 - T\frac{||w||}{2}||w|| \\
&= \frac{||w||}{2}.
\end{aligned}$$

Since v and w are orthogonal, this implies that at least one of $x(t_0)$ and $x(t_1)$ has distance greater than $||w||/4$ from the place $\{x \in \mathbb{R}^n : \langle x, w\rangle = 0\}$ and hence from Ω. Since t_0 (and hence also t_1) can be arbitrarily large, this contradicts Proposition 5.6 in the textbook. Hence we have proven that indeed $F(\lambda v) = 0$ for all $\lambda \in (0, r)$.

.3. If $\bar{x}$ is not an equilibrium point, there exists a transversal line segment S w.r.t. F through $\bar{x}$. Choose a neighborhood U of $\bar{x}$ as in Lemma 5.12. Since $\bar{x}$ is both an α- and an ω-limit point of the given trajectory $x(t)$, it intersects U for times $s_k^{\pm} \to \pm\infty$, $k \to \infty$. Applying Lemma 5.12 we find times $t_j, j \in \mathbb{Z}$ such that $\dots t_{-1} < t_0 < t_1 < t_2 < \dots$, $\lim_{j\to\pm\infty} t_j = \pm\infty$, $x(t_j) \in S$ and $\lim_{j\to\pm\infty} t(x_j) = \bar{x}$. Now it was discussed in class that the pairwise distinct points $x(t_j)$ are monotonically ordered on S, i.e. for $j < k < \ell$ in $\mathbb{Z}$, the point $x(t_k)$ lies between the points $x(t_j)$ and $x(t_\ell)$. This is clearly contradicting $\lim_{j\to\pm\infty} t(x_j) = \bar{x}$, so $\bar{x}$ has to be an equilibrium point.

.5. (i) It is straightforward to check that the vector field defined by F is inward-pointing at the boundary of the square $[-1, 1] \times [-1, 1]$, which implies the assertion.

(ii) Following the hint, we consider the function $f(t) = (1 - x_1(t)^2)(1 - x_2(t)^2)$. A simple calculation using the ODE at hand shows that $f'(t) = -4(x_1(t)^2 + x_2(t)^2)f(t)$, hence $\log(f(t)) = -\int_0^t 4||x(s)||^2 ds$ and thus $f(t) = \exp(-\int_0^t 4||x(s)||^2 ds)$. This first shows that f is monotonically decreasing, hence for all times s we get

$$\frac{3}{4} = f(\frac{1}{2}, 0) \geq f(x(s)) = (1 - x_1(s)^2)(1 - x_2(s)^2) \geq 1 - ||x(s)||^2,$$

hence $||x(s)|| \geq 1/2$ and thus $f(t) = \exp(-\int_0^t 4||x(s)||^2 ds) \to 0$. Since f is continuous and $f^{-1}(0) = \partial Q$, this implies that the ω-limit set is contained in ∂Q.

(iii) If Ω consisted of a single point $\bar{x}$, this point would be an equilibrium point. The only equilibrium points on ∂Q are the four corners $(\pm 1, \pm 1)$, as is obvious from the formula. The differential of F such that $x'(t) = F(x(t))$ can be easily calculated to be

$$DF(x_1, x_2) = \begin{bmatrix} (1 - x_1^2) - 2x_1(x_1 + 2x_2) & 2(1 - x_1)^2 \\ -2(1 - x_1^2) & (1 - x_2^2) - 2x_2(x_2 - 2x_1) \end{bmatrix},$$

hence we see that for $(x_1, x_2) = (\pm 1, \pm 1)$, this matrix is diagonal, hence the eigenvectors are given by e_1 and e_2, and the stable manifold at all these four points can then be seen to agree with the affine subspace spanned by one of these vectors in all cases. Since the trajectory $x(t)$ does not reach ∂Q in finite time (as, by the formula derived above, $f(t)$ is positive for all t), the stable manifold theorem then implies that $f(t)$ cannot converge towards any of these points, hence Ω does not consist of a single point.

(iv) Ω is invariant under the flow under which each of the four edges of the square forms a single trajectory. This implies the assertion.

(v) Suppose Ω does not contain all four edges of the square. Then, suing part (iv), we may assume without loss of generality that $[-1, 1] \times \{-1\} \subset \Omega$ and $\{-1\} \times [-1, 1] \cap \Omega \subset \{(-1, -1), (-1, 1)\}$. Then after applying the affine transformation $(x, y) \to (x + 1, y + 1)$, the assumptions of problem 5.2 are satisfied with $r = 2$ and $v = e_1$. But this would imply that any of the points of the lower edge $[-1, 1] \times \{-1\} \subset \Omega$ are equilibrium points, which is not the case. Hence we have reached a contradiction, so the proof for $\Omega = \partial Q$ is complete.