KEPLER'S LAWS

In the 1600s, Kepler, after analyzing the the astronomical observations of Tycho Brahe, formulated the following laws about the motion of the planets around the sun:

(1) The orbit of each planet is an ellipse with the sun at one focus.
(2) The line segment from the sun to the planet sweeps out equal areas in equal times. (That is, it sweeps out area at a constant rate).
(3) The square of the period is proportional to the cube of the major axis of the ellipse.

In 1687, Newton deduced Kepler's Laws from his general laws of motion and his law of gravitation attraction.

Here, we give such a derivation, following ideas of Simon Kochen (as presented in *Differential Equations* by Simmons and Kranz).

We will ignore the relatively small motion of the sun and pretend that it is fixed at the origin. We will also treat the sun and the planet as point masses. (Actually, Newton took some care to justify that: he showed that if the bodies in question are spherically symmetric about their centers, then they move exactly as the would if they were point masses.) Of course we are also ignoring the forces that other planets, moons, etc exert on a given planet, since those are much smaller than the force exerted by the sun.

Let $f(t) \in \mathbf{R}^3$ be the position of the planet at time t and let m be its mass. The equation of motion is

$$mf'' = -GMm\frac{f}{|f|^2}$$

or

$$f'' = -GM\frac{f}{|f|^3}.$$

Theorem 1. $f \times f' = \mathbf{c}$ *for some constant vector* $\mathbf{c} \in \mathbf{R}^3$.

Proof.

$$(f \times f')' = f' \times f' + f \times f''.$$

Of course $f' \times f' = 0$, and $f \times f'' = 0$ because f'' is a scalar multiple of f. $\square$

Of course the theorem is equivalent to the statement that the angular momentum $f \times mf'$ is constant.

Corollary 2 (Kepler's 2nd Law)**.** *If* $A(t)$ *is the area swept out in the time interval* $[0, t]$*, then* $A'(t) = \frac{1}{2}|\mathbf{c}|$.

This is because the area swept out during $[t, t + \Delta t]$ is approximately half the area of the parallelogram with sides f and $f'\Delta t$, i.e., $\frac{1}{2}|f \times f'\Delta t|$.

Remark 3. Note that the theorem and its corollary hold for any "central" force, i.e., any force that at each time is a scalar multiple of the position vector. (Newton already pointed out this out.)

Elliptic Orbits (Kepler's First Law)

Let $\mathbf{u} = \frac{f}{|f|}$ and $r = |f|$. Thus $f = r\mathbf{u}$. Since $1 \equiv |\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u}$,

$$0 = (\mathbf{u} \cdot \mathbf{u})' = 2\mathbf{u} \cdot \mathbf{u}'.$$

Thus $\mathbf{u} \perp \mathbf{u}'$.

Note that

$$\begin{aligned} \mathbf{c} &= f \times f' \\ &= r\mathbf{u} \times (r'\mathbf{u} + r\mathbf{u}') \\ &= r^2 \mathbf{u} \times \mathbf{u}'. \end{aligned} \tag{1}$$

Thus (since $\mathbf{u} \perp \mathbf{u}'$),

$$|\mathbf{c}| = r^2 |\mathbf{u}||\mathbf{u}'| = 2|\mathbf{u}'|.$$

If $\mathbf{c} = 0$, then $|\mathbf{u}'| = 0$, so $\mathbf{u}$ is constant. Thus in this case, the planet moves in a straight line through the origin. Conservation of energy (potential plus kinetic) gives a first order ODE that one can easily solve.

Thus from now on we will assume that $\mathbf{c} \neq 0$. We have

$$\begin{aligned} f'' \times \mathbf{c} &= f'' \times (f \times f') \\ &= \left(-GM \frac{\mathbf{u}}{r}^2 \right) \times (r^2 \mathbf{u} \times \mathbf{u}') \quad \text{(by (1))} \\ &= -GM\mathbf{u} \times (\mathbf{u} \times \mathbf{u}'). \end{aligned} \tag{2}$$

Note that if $\mathbf{u}$ is a unit vector and $\mathbf{v}$ is perpendicular to $\mathbf{u}$, then $\mathbf{u} \times (\mathbf{u} \times \mathbf{v}) = -\mathbf{v}$. One way to see this: rotate the coordinate system so that $\mathbf{u} = \mathbf{i}$ and $\mathbf{v} = |\mathbf{v}|\,\mathbf{j}$. Then

$$\begin{aligned} \mathbf{u} \times (\mathbf{u} \times \mathbf{v}) &= \mathbf{i} \times (\mathbf{i} \times |\mathbf{v}|\,\mathbf{j}) \\ &= \mathbf{i} \times |\mathbf{v}|\,\mathbf{k} \\ &= -|\mathbf{v}|\,\mathbf{j} \\ &= -\mathbf{v}. \end{aligned}$$

Thus (2) becomes

$$f'' \times \mathbf{c} = GM\mathbf{u}'.$$

Integrating gives

$$f' \times \mathbf{c} = GM(\mathbf{u} + \mathbf{K})$$

for some constant vector $\mathbf{K}$. Let's take the inner product with f:

$$\begin{aligned} f \cdot (f' \times \mathbf{c}) &= f \cdot GM(\mathbf{u} + \mathbf{K}) \\ &= r\mathbf{u} \cdot GM(\mathbf{u} + \mathbf{K}) \\ &= rGM(1 + \mathbf{u} \cdot \mathbf{K}). \end{aligned}$$

Using the identity $A \cdot (B \times C) = \det(A, B, C) = \det(C, A, B) = C \cdot (A \times B)$, we have

$$\mathbf{c} \cdot (f \times f') = rGM(1 + \mathbf{u} \cdot \mathbf{K}).$$

But $f \times f' = \mathbf{c}$, so

$$|\mathbf{c}|^2 = rGM(1 + \mathbf{u} \cdot \mathbf{K}).$$

Let $k = |\mathbf{K}|$ and let θ be the angle between $\mathbf{u}$ and $\mathbf{K}$, so that $\mathbf{u} \cdot \mathbf{K} = |\mathbf{u}|\,|\mathbf{K}|\,\cos\theta = k\cos\theta$. Then

$$|\mathbf{c}|^2 = rGM(1 + k\cos\theta). \tag{3}$$

This is the equation in polar coordinates of a conic section with eccentricity k and with one focus at the origin. If $k = 0$, it's a circle. If $0 < k < 1$, it's an ellipse. If $k = 1$, it's a parabola. If $k > 1$, it's a hyperbola. Thus $k < 1$ corresponds to closed orbits (i.e., to periodic solutions.)

Thus we have proved Kepler's First Law.

Periods

For an ellipse with one focus at the origin, let r_{min} be the minimum distance from the origin to a point on the ellipse and r_{max} be the maximum distance. If a is the semimajor axis (i.e., half the length of the longest chord in the ellipse), then a is clearly the arithmetic mean of r_{min} and r_{max}:

$$a = \frac{1}{2}(r_{\text{min}} + r_{\text{max}}).$$

There is also a simple formula for the semiminor axis b (half the length of the longest chord that is perpendicular to the chord of length $2a$): b is the geometric mean of r_{min} and r_{max}, namely

$$b = \sqrt{r_{\text{min}} r_{\text{max}}}.$$

From (3), we see that

$$r_{\text{min}} = \frac{|\mathbf{c}|^2}{GM(1+k)}, \qquad r_{\text{max}} = \frac{|\mathbf{c}|^2}{GM(1-k)},$$

so

$$a = \frac{1}{2}(r_{\text{min}} + r_{\text{max}}) = \frac{|\mathbf{c}|^2}{GM(1-k^2)} \tag{4}$$

$$b = \sqrt{r_{\text{min}}\, r_{\text{max}}} = \frac{|\mathbf{c}|^2}{GM\sqrt{1-k^2}}. \tag{5}$$

Hence

$$b = a\sqrt{1-k^2}.$$

Consequently, the area inside the ellipse is

$$\mathcal{A} = \pi ab = \pi a^2\sqrt{1-k^2}.$$

Recall that $\frac{1}{2}|\mathbf{c}|$ is the rate at which area is swept out. Thus if T is the period, then

$$\left(\frac{1}{2}|\mathbf{c}|\right) T = \mathcal{A} = \pi a^2\sqrt{1-k^2},$$

so

$$T^2 = \frac{4\pi^2 a^4(1-k^2)}{|\mathbf{c}|^2}.$$

By (4),

$$|\mathbf{c}|^2 = aGM(1-k^2).$$

Thus

$$T^2 = \frac{4\pi^2 a^3}{GM},$$

which is Kepler's Third Law.