LYAPUNOV AND LA SALLE/KRASOVSKI THEOREMS

Definition 1. Suppose that p is an equilibrium point of the autonomous system $x'(t) = F(x(t))$, i.e., that $F(p) = 0$. We say that p is **stable** if, given any real number $r > 0$, there is an open set W containing p such that $\phi_t(x) \in \mathbf{B}_r(x)$ for all $x \in W$ and $t \geq 0$. We say that p is **asymptotically stable** if it is stable and if there is an open neighborhood G of x such that $\lim_{t\to\infty} \phi_t(x) = p$ for all $x \in G$.

(This is equivalent to the definition in the text, thought it is worded slightly differently.)

Theorem 2 (Lyapunov's First Theorem)**.** *Suppose that $U \subset \mathbf{R}^n$ is an open set, that $F : U \subset \mathbf{R}^n \to \mathbf{R}^n$ is a C^1 vectorfield, and that p is a point in U such that $F(p) = 0$. Suppose that $R > 0$, that $\overline{\mathbf{B}_R(p)} \subset U$, and that V is a C^1 function on $\overline{\mathbf{B}_R(p)} \subset U$ such that*

1. *$V(p) = 0 < V(x)$ for all $x \neq p$ in $\overline{\mathbf{B}_R(0)}$.*
2. *$\nabla V \cdot F \leq 0$ at all points of $\overline{\mathbf{B}_R(0)}$.*

Then p is a stable equilibrium for $x' = F(x)$.

Proof. It suffices to prove it for $p = 0$. Note that if $x \in U$ and $I \subset \mathbf{R}$ is an interval such that $\phi_t(x)$ exists and is in $\overline{\mathbf{B}_R(0)}$ for all $t \in I$, then for $t \in I$,

$$\begin{aligned} \frac{d}{dt} V(\phi_t(x)) &= \nabla V(\phi_t(x)) \cdot \frac{d}{dt}\phi_t(x) \\ &= \nabla V(\phi_t(x)) \cdot F(\phi_t(x)) \\ &\leq 0, \end{aligned}$$

and thus

$$t \in I \mapsto V(\phi_t(x)) \text{ is a decreasing function.} \tag{1}$$

Let $0 < r \leq R$. Let $\eta = \min\{V(x) : r \leq x \leq R\}$. Then $\eta > 0$ by hypothesis. Let

$$W = \{x \in \mathbf{B}_R(0) : V(x) < \eta\}$$

Let $x \in \overline{W}$. Let (a_x, b_x) be the maximal interval on which $\phi_t(x)$ exists and is in $\mathbf{B}_R(0)$.

By (1),

$$V(\phi_t(x)) \leq V(\phi_0(x)) = V(x) \leq \eta/2 \quad \text{for all } t \in [0, b_x).$$

Thus $\phi_t(x) \in \mathbf{B}_r(0)$ for all $t \in [0, b_x)$. Hence by Theorem ?, $b_x = \infty$.

We have shown: if $x \in W$, then $b_x = \infty$ and $\phi_t(x) \in \mathbf{B}_r(0)$ for all $t \geq 0$. Since W is an open set that contains 0, 0 is stable. □

Theorem 3 (La Salle/Krasovski Theorem)**.** *Suppose that $U \subset \mathbf{R}^n$ is an open set, that $F : U \subset \mathbf{R}^n \to \mathbf{R}^n$ is a C^1 vectorfield, and that p is a point in U such that $F(p) = 0$. Suppose that $R > 0$, that $\overline{\mathbf{B}_R(p)} \subset U$, and that V is a C^1 function on $\overline{\mathbf{B}_R(p)}$ such that*

1. *$V(p) = 0 < V(x)$ for all $x \neq p$ in $\overline{\mathbf{B}_R(0)}$.*

(2) $\nabla V \cdot F \le 0$ at all points of $\overline{\mathbf{B}_R(0)}$.

Let

$$K = \{p \in \overline{\mathbf{B}(p,R)} : \nabla V(p) \cdot F(p) = 0\}.$$

Suppose also that

$$\text{(*)} \qquad \text{If } \phi_t(x) \in K \text{ for all } t \ge 0, \text{ then } x = p.$$

Then p is an asymptotically stable equilibrium for $x' = F(x)$.

Note that the hypotheses are the hypotheses of Lyapunov's First Theorem, plus the hypothesis (*).

Proof. It suffices to prove it when $p = 0$. By Lyapunov's First Theorem, there is an open set W containing 0 such that for all $x \in W$ and $t \ge 0$, $\phi_t(x) \in \mathbf{B}_R(p)$. Exactly as in the proof of that theorem, $V(\phi_t(x))$ is a decreasing function of t for $t \ge 0$. Thus it has a limit as $t \to \infty$:

$$\lim_{t\to\infty} V(\phi_t(x)) = L. \tag{2}$$

Let

$$r = \limsup_{t\to\infty} |\phi_t(x)|. \tag{3}$$

By definition of lim sup, there is a sequence $t_i \to \infty$ such that

$$|\phi_{t_i}(x)| \to r. \tag{4}$$

By Bolzano-Weiestrass, there is a subsequence $\phi_{t_{i(j)}}(x)$ that converges to a limit q. By relabeling, we can assume that the original sequence converges to q:

$$\phi_{t_i}(x) \to q. \tag{5}$$

Since ϕ is continuous,

$$\phi_t(\phi_{t_i}(x)) \to \phi_t(q) \quad \text{as } i \to \infty$$

for all $t \ge 0$. That is,

$$\phi_{t+t_i}(x) \to \phi_t(q).$$

Hence

$$V(\phi_{t+t_i}(x)) \to V(\phi_t(q)).$$

But the left hand side converges to L as $i \to \infty$ (by (2)). Thus

$$L = V(\phi_t(q)) \quad \text{for all } t \ge 0.$$

Hence

$$\begin{aligned} 0 &= \frac{d}{dt}(V(\phi_t(q)) \\ &= \nabla V(\phi_t(q)) \cdot \frac{d}{dt}\phi_t(q) \\ &= \nabla V(\phi_t(q)) \cdot F(\phi_t(q)). \end{aligned}$$

Thus $\phi_t(q) \in K$ for all $t \ge 0$.

By the assumption (*), $q = 0$. Thus $r = 0$ (by (4) and (5)), and so $\phi_t(x) \to 0$ by (3).

We have proved that if $x \in W$, then $\lim_{t\to\infty} \phi_t(x) = 0$. Thus 0 is asymptotically stable. □

Theorem 4 (Lyapunov's Second Theorem). *Suppose that $U \subset \mathbf{R}^n$ is an open set, that $F : U \subset \mathbf{R}^n \to \mathbf{R}^n$ is a C^1 vectorfield, and that p is a point in U such that $F(p) = 0$. Suppose that $R > 0$, that $\overline{\mathbf{B}_R(p)} \subset U$, and that V is a C^1 function on $\overline{\mathbf{B}_R(p)}$ such that*

(1) *$V(p) = 0 < V(x)$ for all $x \neq p$ in $\overline{\mathbf{B}_R(p)}$.*
(2) *$\nabla V \cdot F < 0$ at all points of $\overline{\mathbf{B}_R(p)} \setminus \{p\}$.*

Then p is an asymptotically stable equilibrium for $x' = F(x)$.

Proof. As in La Salle's Theorem, let

$$K = \{q \in \overline{\mathbf{B}_R(p)} : \nabla V(q) \cdot F(q) = 0\}.$$

By (2), $K = \{p\}$. This implies that hypothesis (*) of La Salle's Theorem holds.

Thus 0 is asymptotically stable by La Salle's Theorem. □

Examples

Example 1. Consider the system

$$\begin{aligned} x' &= -y - x^3, \\ y' &= x - y^3, \end{aligned}$$

i.e.,

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -x^3 \\ -y^3 \end{bmatrix}$$

Let $V(x, y) = x^2 + y^2$. Clearly V has its minimum at $(0, 0)$ and nowhere else. Note that

$$\begin{aligned} \nabla V \cdot F &= (2x, 2y) \cdot (-y - x^3, x - y^3) \\ &= -2x^4 - 2y^4, \end{aligned}$$

which is < 0 except at the point $(x, y) = (0, 0)$.

By Lyapunov's Second Theorem, the origin is an asymptotically stable equilibrium.

Example 2. Now consider the system

$$\begin{aligned} x' &= -y - x^3, \\ y' &= x \end{aligned}$$

i.e.,

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -x^3 \\ 0 \end{bmatrix}.$$

Again, let's try $V(x, y) = x^2 + y^2$. Then

$$\begin{aligned} \nabla V \cdot F &= (2x, 2y) \cdot (-y - x^3, x) \\ &= -2x^4. \end{aligned}$$

This is ≤ 0 everywhere, so the origin is a stable equilibrium by Lyapunov's First Theorem.

Note that

$$\nabla V(x, y) = 0 \iff x = 0.$$

This means we cannot apply Lyapunov's Second Theorem: no matter how small $R > 0$ is, $\overline{\mathbf{B}_R(0)}$ will contain many points (namely all points $(x, y) \in \overline{\mathbf{B}_R(0)}$ with $x = 0$) where $\nabla V \cdot F = 0$.

But we can apply La Salle's Theorem. Let

$$K = \{(x, y) : \nabla V(x, y) \cdot F(x, y) = 0\}.$$

Then (as we just calculated)

$$K = \{(x, y) : x = 0\}.$$

Suppose $(x(t), y(t))$ is a solution of the ODE such that $(x(t), y(t)) \in K$ for all $t \geq 0$. Then $x(t) \equiv 0$, so

$$\begin{aligned} 0 &= x'(t) \\ &= -y(t) - x(t)^3 \\ &= -y(t) \end{aligned}$$

Thus $y(t) = 0$ for all $t \geq 0$.

We have shown: the only solution that stays in K for all $t \geq 0$ is $(x(t), y(t)) \equiv (0, 0)$.

Hence by La Salle, $(0, 0)$ is asymptotically stable.