PEANO EXISTENCE THEOREM

BRIAN WHITE

Theorem 1. *Suppose that U is an open subset of $\mathbf{R}^N$, that $x_0 \in U$, and that $F : U \to \mathbf{R}^N$ is continuous. Then there exists a $\delta > 0$ and a continuously differentiable function $x : [0, \delta] \to U$ such that*

$$\begin{aligned} &x(0) = x_0, \text{ and} \\ &x'(t) = F(x(t)) \quad \text{for } t \in [0, \delta]. \end{aligned}$$

Proof. **Case 1**: $U = \mathbf{R}^N$ and F is bounded: $\sup_x |F(x)| \le M < \infty$.

For $k \in \mathbf{N}$, define $x_k : [0, \infty) \to \mathbf{R}^N$ by

$$x_k(t) = \begin{cases} x_0 + tF(x_0) & \text{for } 0 \le t \le \frac{1}{k}, \\ x(\frac{1}{k}) + (t - \frac{1}{k})F(x(\frac{1}{k})) & \text{for } \frac{1}{k} \le t \le \frac{2}{k}, \\ x(\frac{2}{k}) + (t - \frac{2}{k})F(x(\frac{2}{k})) & \text{for } \frac{2}{k} \le t \le \frac{3}{k}, \\ \dots \end{cases}$$

That is, we define x_k inductively on $[\frac{j}{k}, \frac{j+1}{k}]$ by

$$x_k(t) = x_k(\tfrac{j}{k}) + (t - \tfrac{j}{k})F(x_k(\tfrac{j}{k})) \quad \text{for } \tfrac{j}{k} \le t \le \tfrac{j+1}{k}.$$

We also define a piecewise-constant function $y_k : [0, \infty) \to \mathbf{R}^N$ by

$$y_k(t) = x_k(\tfrac{j}{k}) \quad \text{for } \tfrac{j}{k} \le t < \tfrac{j+1}{k}. \tag{1}$$

For $t \in [0, \infty)$, let $\frac{j}{k} \le t < \frac{j+1}{k}$. Then

$$x_k'(t) = F(x_k(\tfrac{j}{k}) = F(y_k(t))$$

by (1), so

$$x_k(t) = x_0 + \int_{s=0}^{t} F(y_k(s))\, ds. \tag{2}$$

Claim 1. $|x_k(t) - x_k(\tau)| \le M|t - \tau|$.

Proof of Claim 1. By (2),

$$\begin{aligned} |x_k(t) - x_k(\tau)| &= \left| \int_{s=\tau}^{t} F(y_k(s))\, ds \right| \\ &\le \int_{s=\tau}^{t} |F(y_k(s))|\, ds \\ &\le \int_{\tau}^{t} M\, ds \\ &= M|t - \tau|. \end{aligned}$$

∎

Claim 2. *The $x_k(\cdot)$ are uniformly bounded on $[0,1]$:*

$$\sup_k \sup_{t\in[0,1]} |x_k(t)| < \infty.$$

Proof of Claim 2. By Claim 1,

$$\begin{aligned} |x_k(t) - x_0| &= |x_k(t) - x_k(0)| \\ &\le Mt \end{aligned}$$

so

$$x_k([0,1]) \subset \overline{\mathbf{B}_M(x_0)}, \tag{3}$$

and therefore

$$\sup_{t\in[0,1]} |x_k(t)| \le M + |x_0|.$$

□

According to the Arzela-Ascoli Theorem, Claims 1 and 2 imply that the sequence $x_k : [0,1] \to \mathbf{R}^N$ has a uniformly convergent subsequence $x_{k(i)} : [0,1] \to \mathbf{R}^N$. That is, there is a function $x : [0,1] \to \mathbf{R}^N$ such that

$$\sup_{t\in[0,1]} |x_{k(i)}(t) - x(t)| \to 0. \tag{4}$$

Since $x(\cdot)$ is a uniform limit of continuous functions, it is also continuous.

Note that $x_k(t)$ and $y_k(t)$ are very close:

$$\begin{aligned} |x_k(t) - y_k(t)| &= |x_k(t) - x_k(\tfrac{j}{k})| \quad (\text{where } \tfrac{j}{k} \le t < \tfrac{j+1}{k}) \\ &\le M|t - \tfrac{j}{k}| \\ &\le \frac{M}{k}. \end{aligned} \tag{5}$$

Thus

$$\begin{aligned} |y_{k(i)}(t) - x(t)| &\le |y_{k(i)}(t) - x_{k(i)}(t)| + |x_{k(i)}(t) - x(t)| \\ &\le \frac{M}{k(i)} + |x_{k(i)}(t) - x(t)| \end{aligned}$$

so

$$\begin{aligned} \sup_{t\in[0,1]} |y_{k(i)}(t) - x(t)| &\le \frac{M}{k(i)} + \sup_{t\in[0,1]} |x_{k(i)}(t) - x(t)| \\ &\to 0 \end{aligned} \tag{6}$$

by (4). That is, the $y_{k(i)}(\cdot)$ also converge uniformly to $x(\cdot)$ on $[0,1]$.

Claim 3. *$F(y_{k(i)}(\cdot))$ converges uniformly to $F(x(\cdot))$ on $[0,1]$.*

Proof. Note that $y_{k(i)}([0,1]) \subset x_{k(i)}([0,1])$ so $y_{k(i)}([0,1]) \subset \overline{\mathbf{B}_M(x_0)}$ by (3).

Since F is continuous and $\overline{\mathbf{B}_M(x_0)}$ is compact, F is uniformly continuous on $\overline{\mathbf{B}_M(x_0)}$. Let $\epsilon > 0$. Then there is a $\delta > 0$ such that

$$p, q \in \overline{\mathbf{B}_M(x_0)},\ |p - q| \le \delta \implies |F(p) - F(q)| < \epsilon.$$

By the uniform convergence $y_{k(i)}(\cdot) \to x(\cdot)$,

$$\sup_{t\in[0,1]} |y_{k(i)}(t) - x(t)| \le \delta$$

for all sufficiently large i. Thus

$$\sup_{t\in[0,1]} |F(y_{k(i)}(t) - F(x(t))| \le \epsilon$$

for all sufficiently large i. This proves Claim 3. □

Recall (see (2)) that

$$x_{k(i)}(t) = x_0 + \int_{s=0}^{t} F(y_{k(i)}(s))\, ds.$$

Letting $i \to \infty$ gives

$$x(t) = x_0 + \int_{s=0}^{t} F(x(s))\, ds$$

by (4) and (3). Thus $x(0) = x_0$, and by differentiating, we see that

$$x'(t) = F(x(t)).$$

This completes the proof in Case 1.

Case 2: General $U \subset \mathbf{R}^N$ and continuous $F : U \to \mathbf{R}^N$.

Let $\overline{\mathbf{B}_R(x_0)}$ be compact ball contained in U. For $x \in \mathbf{R}^N$, let $\Pi(x)$ be the point in $\overline{\mathbf{B}_R(x_0)}$ closest to x:

$$\Pi(x) = \begin{cases} x & \text{if } x \in \overline{\mathbf{B}_R(x_0)}, \text{ and} \\ x_0 + R\frac{x-x_0}{|x-x_0|} & \text{if not.} \end{cases}$$

Define $\widehat{F} : \mathbf{R}^N \to \mathbf{R}^N$ by

$$\widehat{F}(x) = F(\Pi(x)).$$

Then $\widehat{F}$ is continuous and bounded, so by Case 1, there is a differentiable function

$$x : [0,1] \to \mathbf{R}^N$$

such that

$$x(0) = x_0,$$
$$x'(t)\widehat{F}(x(t)).$$

For some small $\delta > 0$, $x(t) \in \overline{\mathbf{B}_R(x_0)}$ for all $t \in [0, \delta]$. For such t,

$$x'(t) = \widehat{F}(x(t)) = F(x(t)).$$

□