FINDING THE MATRIX EXPONENTIAL

(1) Find the characteristic polynomial $\det(\lambda I - A)$.
(2) Factor it:
$$\det(\lambda I - A) = \prod_{i=1}^{k} (\lambda - \lambda_i)^{\nu_i}.$$
(3) For each i, solve
$$(\lambda_i I - A)^{\nu_i} \mathbf{v} = 0$$
(e.g., by Gauss Elimination) to get a basis $\mathcal{B}_i$ of $\ker(\lambda_i I - A)^{\nu_i}$.
(4) Let $\mathcal{B}$ be the basis consisting of the vectors in $\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_k$ (listed in that order). Let S be the matrix whose columns are the basis vectors in $\mathcal{B}$. Then $S^{-1}AS$ is in block diagonal form:
$$S^{-1}AS = B = \begin{bmatrix} B_1 & 0 & \dots & 0 \\ 0 & B_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_k \end{bmatrix}.$$

Here block B_i is a $\nu_i \times \nu_i$ matrix with characteristic polynomial $(\lambda - \lambda_i)^{\nu_i}$, and
$$(B_i - \lambda_i I)^{\nu_i} = 0.$$
Thus if we write $\mathcal{N}_i = B_i - \lambda_i I$, then
$$\begin{aligned} e^{tB_i} &= e^{t\lambda_i I + t\mathcal{N}_i} \\ &= e^{t\lambda_i I} e^{t\mathcal{N}_i} \qquad \text{(since } t\lambda_i I \text{ and } \mathcal{N}_i \text{ commute)} \\ &= e^{\lambda_i t} \left(I + t\mathcal{N}_i + \frac{t^2}{2!}\mathcal{N}_i^2 + \dots + \frac{t^{\nu_i - 1}}{(\nu_i - 1)!}\mathcal{N}_i^{\nu_i - 1} \right). \end{aligned} \tag{*}$$
So we have found e^{tB_i} (and therefore e^{tB}). Now
$$e^{tA} = S e^{tB} S^{-1} = S \begin{bmatrix} e^{tB_1} & 0 & \dots & 0 \\ 0 & e^{tB_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{tB_k} \end{bmatrix} S^{-1}.$$
(where the e^{tB_i} are given by (*).)

Of course if A is large, it might not be possible to factor the characteristic polynomial. For such A, one would generally approximate e^{tA} by computer.

The $L + N$ Decomposition

The book emphasizes that any $n \times n$ complex matrix A can be written as the sum of a diagonalizable matrix L and a nilpotent matrix N. **Note that we did not need this fact to compute e^{tA} above**.

However, if you want to find L and N, here is how you can do it:

Above, we found a decomposition of $B = S^{-1}AS$ into a **diagonal** matrix D and a nilpotent matrix $\mathcal{N}$, namely

$$(\dagger) \qquad B = S^{-1}AS = D + \mathcal{N} = \begin{bmatrix} \Lambda_1 & 0 & \dots & 0 \\ 0 & \Lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Lambda_k \end{bmatrix} + \begin{bmatrix} \mathcal{N}_1 & 0 & \dots & 0 \\ 0 & \mathcal{N}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathcal{N}_k \end{bmatrix}$$

where Λ_i is the diagonal $\nu_i \times \nu_i$ matrix with λ_i on the diagonal:

$$\Lambda_i = \lambda_i I_{\nu_i \times \nu_i}$$

and $\mathcal{N}_i = B_i - \Lambda_i$. (Here $I_{\nu_i \times \nu_i}$ denotes the $\nu_i \times \nu_i$ identity matrix.)

If we multiply ($\dagger$) on the left by S and on the right by S^{-1}, we get

$$A = SDS^{-1} + S\mathcal{N}S^{-1}.$$

Now SDS^{-1} is diagonalizable and $S\mathcal{N}S^{-1}$ is nilpotent, so we have found the $L+N$ decompostion: $L = SDS^{-1}$ and $N = S\mathcal{N}S^{-1}$.