SAMPLE TEST PROBLEMS

You can expect a number of problems similar to hw problems. Here are a few additional sample problems.

0. Of course, given A, you should be able to compute e^{tA}. Since time is limited, such A will probably be small. Also, to save time, you'll probably be allowed to leave your answer in the form $S \dots S^{-1}$ rather than having to multiple out the matrices.

1. Let A be an $n \times n$ complex matrix. Show that if λ is an eigenvalue of A, then $|\lambda| \le \|A\|_{\text{op}}$.

2. Let A be an $n \times n$ matrix and $f : \mathbf{R} \to \mathbf{C}^n$ be a continuous function. Derive the formula for the solution to the initial value problem

$$\begin{aligned} \mathbf{x}' &= A\mathbf{x} + f(\mathbf{x}) \\ \mathbf{x}(0) &= \mathbf{x}_0. \end{aligned}$$

3. If A is an $n \times n$ matrix, prove that $\|e^A\|_{\text{op}} \le e^{\|A\|_{\text{op}}}$.

4. Suppose A is an $n \times n$ matrix. Suppose the characteristic polynomial $\det(\lambda I - A)$ has a nonzero purely imaginary root. Prove that $\mathbf{x}' = A\mathbf{x}$ has a nonzero periodic solution.

5. Suppose $F_k : \mathbf{R}^n \to \mathbf{R}^n$ is a sequence of smooth functions such that

$$|F_k(x)| \le K < \infty$$

for all $x \in \mathbf{R}^n$ and all $k = 1, 2, \dots$, and such that $F_k \to F$ uniformly.

Suppose $x_k : [0, 1] \to \mathbf{R}^n$ is a sequence sequence of C^1 functions such that

$$\begin{aligned} \mathbf{x}_k'(t) &= F_k(\mathbf{x}_k(t)) \quad (t \in [0, 1]), \\ \mathbf{x}_k(0) &= 0. \end{aligned}$$

Prove that there is a subsequence $\mathbf{x}_{k(i)}(\cdot)$ such that $\mathbf{x}_{k(i)}(\cdot)$ converges uniformly to a limit $\mathbf{x}(\cdot)$ and that

$$\mathbf{x}'(t) = F(\mathbf{x}(t)) \quad (t \in [0, 1]).$$

6. Find the solution to

$$\begin{aligned} x'(t) &= (\cos \tfrac{2t}{\pi})(1 + x(t))^2, \\ x(0) &= 0 \end{aligned}$$

where $x(\cdot)$ is a real-valued function. In addition, determine its maximal interval of existence.

7. Find all solutions of $y' + \frac{1}{t}y = t$ on the interval $t \in (0, \infty)$.