Math 63CM discussion section problems for April 5, 2019.

These problems are *not to be turned in*, but are provided in the hope that you will find some of them interesting and instructive. Some are routine, and some are quite challenging. Feel free to come to office hours if you want to discuss any of them beyond what we have time for in section.

1. Consider the initial-value problem
$$\begin{cases} \dot{x} = x^{2/3} \\ x(0) = x_0. \end{cases} \tag{1}$$
   (a) Find infinitely many solutions to (1) with $x_0 = 0$.

   (b) Prove that if $x_0 > 0$, then there is a unique solution (depending on x_0) to (1).

   (c) If you solve (1) with $x_0 = 0$ by the "separation of variables" method, you will obtain a single solution, and not any of the other infinitely many solutions you found in 1a. What is "wrong" with the method?

2. Consider the initial-value problem
$$\begin{cases} \dot{x} = x(1-x)(1+x) \\ x(0) = x_0. \end{cases} \tag{2}$$
   For fixed $t \in (0,\infty)$, is $x(t)$ continuous as a function of x_0? What is $\lim_{t\to\infty} x(t)$? Is $\lim_{t\to\infty} x(t)$ continuous as a function of x_0?

3. Consider the initial-value problem
$$\begin{cases} \dot{x} = a + \sin(x) \\ x(0) = x_0. \end{cases} \tag{3}$$
   Find $\lim_{x\to\infty} x(t)$ as a function of x_0 and a.

4. Consider the initial-value problem
$$\begin{cases} \dot{x} = x^2 \\ x(0) = x_0. \end{cases} \tag{4}$$
   For which x_0 does (4)...

   (a) have unique solutions?

   (b) have solutions which are defined for all time?

5. *The basic ODE framework is pretty general.*

   (a) Consider the *non-autonomous* (time-dependent) ODE
$$\dot{x}(t) = F(t, x(t)).$$
   Can you think of a way to write this as an autonomous ODE of the form
$$\dot{y}(t) = G(y(t))?$$
   (b) Consider a second-order ODE
$$\ddot{x}(t) = F(x(t), \dot{x}(t)).$$
   Can you think of a way to write this as an autonomous ODE of the form
$$\dot{y}(t) = G(y(t))?$$

6. *Hölder-continuous functions.* Let $\Omega \subset \mathbf{R}^d$. We say that a function $f : \Omega \to \mathbf{R}^d$ is *Hölder-α continuous* if
$$\sup_{x\neq y\in\Omega} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < \infty.$$

(a) Prove that if $f : \Omega \to \mathbf{R}^d$ is Hölder-α continuous with $\alpha > 1$, then f is locally constant.

(b) Prove that if $f : \Omega \to \mathbf{R}^d$ is Hölder-α continuous, and $\beta < \alpha$, then f is also Hölder-β continuous.

(c) If $\alpha \in (0,1)$ and $f : \Omega \to \mathbf{R}^d$ is Hölder-α continuous, define

$$\|f\|_{C^\alpha(\Omega)} = \sup_{x\in\Omega} |f(x)| + \sup_{x\neq y\in\Omega} \frac{|f(x)-f(y)|}{|x-y|^\alpha}. \tag{5}$$

Prove that for any $C < \infty$, the set

$$\mathcal{F}_C = \{f : \Omega \to \mathbf{R}^d \mid f \text{ is Hölder-}\alpha \text{ continuous and } \|f\|_{C^\alpha(\Omega)} \leq C\}$$

is equicontinuous.

(d) Prove that if $f \in C^1(\Omega)$, then the definition (5) with $\alpha = 1$ agrees with the definition of the $C^1(\Omega)$ norm that you know:

$$\|f\|_{C^1(\Omega)} = \sup_{x\in\Omega} |f(x)| + \sup_{x\in\Omega} |f'(x)|.$$

(e) Show that if f is locally C^1, then f is locally Lipschitz.

(f) Exhibit a Hölder-1 (i.e., Lipschitz) function f which is not C^1.

(g) For any $\alpha \in (0,1)$, find an $f \in C^\alpha(\mathbf{R})$ and an $x_0 \in \mathbf{R}$ so that the initial-value problem

$$\begin{cases} \dot{x} = f(x) \\ x(0) = x_0 \end{cases}$$

has more than one solution.

7. *Open-ended question*. Take an ODE which does not have unique solutions, and try to solve it numerically using Euler's method. How does the non-uniqueness of solutions manifest numerically? What happens if you use a different numerical method? (You could try implicit Euler, trapezoidal, Adams–Bashforth....)